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MONOTONICITY OF CERTAIN CLASSES OF FUNCTIONS RELATED WITH CUSA — HUYGENS INEQUALITY

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The theory of analytical inequalities is still a very popular field of investigation of many authors. In this note, we analyze the monotonicity of certain classes of functions related with the Cusa — Huygens inequality, and propose some open problems in this direction.

Keywords: *Cusa — Huygens inequality, MTP inequalities.*

1. Introduction

The Cusa — Huygens inequality, stating that

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad x \in (0, \pi/2), \tag{1}$$

is well known in the theory of analytic inequalities (see e.g., the classical monograph [1] by D.S. Mitrinović). This inequality was stated witout a rigorous proof by the German philosopher and theologian Nicolaus de Cusa (1401–1464). The first rigorous proof of Cusa's inequality was given by the Dutch astronomer and physicist Christiaan Huygens (1629–1695). In [2, Remark 4.1], L. Zhu has recently observed that the inequality (1) holds for $x \neq 0$:

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad x \in \mathbb{R} \setminus \{0\}; \tag{2}$$

for more details about the recent generalizations of the Cusa — Huygens inequality, we refer the reader to the research articles [3; 4] and references cited therein.

Let us define, for every $x \in \mathbb{R}$,

$$\Phi_1(x) := \left(\frac{\pi}{2}-1\right)^{-1}(x-\sin x), \quad \Phi_2(x) := \left(\frac{\pi}{2}-1\right)^{-2}(x-\sin x)^2$$

and

$$\Psi_1(x) := \sin x - x\cos x, \quad \Psi_2(x) := (\sin x - x\cos x)^2.$$

In a recent research article [5] by Y. J. Bagul, B. Banjac, C. Chesneau, M. Kostić and B. Malešević (see [5, Theorem 1, Theorem 2]), the following results have been proved:

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Theorem 1. *Let $x \in (0, \pi/2)$. Then we have*

$$\frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right) \Phi_1(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right) \Phi_2(x).$$

Theorem 2. *Let $x \in (0, \pi/2)$. Then we have*

$$\frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right) \Psi_1(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right) \Psi_2(x).$$

In the proof of [5, Theorem 1], we have shown that the function

$$f(x) := \frac{3\sin x - x\cos x - 2x}{x(x - \sin x)}, \quad x \in (0, \pi/2),$$

is strictly monotonically decreasing by applying the well-known l'Hôpital's rule of monotonicity (after that, we have expanded the function $x/\sin x$, $x \in (0, \pi)$ in a Maclaurin series and used a recent result of F. Qi [6] concerning the inequalities for the ratio of two consecutive Bernoulli numbers). Furthermore, in the proof of [5, Theorem 2], we have shown that the function

$$g(x) := \frac{3\sin x - x\cos x - 2x}{x(\sin x - x\cos x)}, \quad x \in (0, \pi/2),$$

is strictly monotonically decreasing by applying the same rule.

This paper aims to investigate the monotonicity of functions

$$f_\alpha(x) := \frac{3\sin x - x\cos x - 2x}{x(x - \sin x)^\alpha}, \quad x > 0,$$

and

$$g_\alpha(x) := \frac{3\sin x - x\cos x - 2x}{x(\sin x - x\cos x)^\alpha}, \quad x \in (0, \pi),$$

depending on the real parameter $\alpha > 0$. The function $f_\alpha(\cdot)$ is well defined and negative due to the fact that $\sin x < x$, $x > 0$, and (2) holds, while the function $g_\alpha(\cdot)$ is well defined and negative due to the fact that $\sin x - x\cos x > 0$, $x \in (0, \pi)$ and (2) holds. Moreover, it can be easily proved that

$$f_\alpha(x) \sim \frac{(-1)}{10 \cdot 6^{\alpha+1}} x^{4-3\alpha}, \quad x \to 0+, \quad g_\alpha(x) \sim \frac{(-1)\cdot 3^{\alpha-1}}{20} x^{4-3\alpha}, \quad x \to 0+.$$

2. The main results, observations and open problems

Let $\alpha > 0$ and $x > 0$. Then a simple computation shows that:

$$f'_\alpha(x) = \frac{x(x-\sin x)(2\cos x + x\sin x - 2) - (3\sin x - x\cos x - 2x)(x - \sin x)}{x^2(x-\sin x)^{\alpha+1}} -$$

$$-\frac{\alpha x(1-\cos x)(3\sin x - x\cos x - 2x)}{x^2(x-\sin x)^{\alpha+1}} := \tag{3}$$

$$:= \frac{z_\alpha(x)}{x^2(x-\sin x)^{\alpha+1}} \sim \frac{6^{\alpha-1}}{10}(3\alpha - 4)x^{3-3\alpha}, \quad x \to 0+. \tag{4}$$

Expanding the function $z_{4/3}(\cdot)$ in a Maclaurin series by using the online calculation www.emathhelp.net/calculators/calculus-1/taylor-and-maclaurin-series-calculator with $n = 10$, we get

$$z_{4/3}(x) = \frac{(-1)}{9450}x^{10} + \cdots, \quad x > 0,$$

so that

$$f'_{4/3}(x) \sim 6^{7/3}\frac{(-x)}{9450}, \quad x \to 0+.$$

Using this estimate and (4), which shows that

$$\lim_{x\to 0+} f'_\alpha(x) = -\infty \text{ for } 0 < \alpha < 4/3 \text{ and } \lim_{x\to 0+} f'_\alpha(x) = +\infty \text{ for } \alpha > 4/3,$$

we get the following:

Theorem 3. *Let $\alpha > 0$. Then there exists a finite real number $x'_\alpha > 0$ such that the following holds:*

(a) *If $0 < \alpha \leq 4/3$, then the function $f_\alpha(\cdot)$ is strictly decreasing on the interval $(0, x'_\alpha)$.*

(b) *If $\alpha > 4/3$, then the function $f_\alpha(\cdot)$ is strictly increasing on the interval $(0, x'_\alpha)$.*

If $0 < \alpha \leq 4/3$, resp. $\alpha > 4/3$, then we define

$$x_\alpha := \sup\{x'_\alpha > 0 : \text{ the function } f_\alpha(\cdot) \text{ is strictly decreasing on } (0, x'_\alpha)\},$$

resp.

$$x_\alpha := \sup\{x'_\alpha > 0 : \text{ the function } f_\alpha(\cdot) \text{ is strictly increasing on } (0, x'_\alpha)\}.$$

Further on, if $x > 0$ and $x \notin 2\mathbb{N}\pi$, then we have

$$f'_\alpha(x) = x^{-1}\big(x - \sin x\big)^{-1-\alpha}(1 - \cos x)(3\sin x - x\cos x - 2x)\big[F(x) - \alpha\big], \qquad (5)$$

where

$$F(x) := \frac{(x - \sin x)\left[x\,(2\cos\ x + x\sin\ x - 2) - (3\sin x - x\cos x - 2x)\right]}{x\,(1 - \cos x)\,(3\sin x - x\cos x - 2x)}.$$

Using the symbolab computation of the first derivative, we get

$$F'(x) = \frac{x\left(-3\sin x + x\sin^2 x - x^2\sin 2x + x^3\cos x + \frac{3\sin 2x}{2} + 3x\cos x - 3x\cos^2 x\right)}{x^2\,(1 - \cos x)^2\,(3\sin x - x\cos x - 2x)^2}\times$$

$$\times\,(1 - \cos x)\,(3\sin x - x\cos x - 2x) -$$

$$-\frac{\left(3\sin x - 4x - x\cos^2 x - x^2\sin 2x - x^2\sin x - \frac{3\sin 2x}{2} + 3x\sin^2 x + 5x\cos x\right)}{x^2\,(1 - \cos x)^2\,(3\sin x - x\cos x - 2x)^2}\times$$

$$\times\,(x - \sin x)\left(x^2\sin x + 3x\cos x - 3\sin x\right) :=$$

$$:= \frac{H(x)}{x^2\,(1 - \cos x)^2\,(3\sin x - x\cos x - 2x)^2}, \quad x \in (0, \infty) \setminus 2\mathbb{N}\pi.$$

It can be simply proved that $\lim_{x\to 0} F(x) = 4/3$, so that the function $F(\cdot)$ can be analytically extended to an open region $\Omega \subseteq \mathbb{C}$ containing the interval $[0, 2\pi)$; we will denote such an analytic extension by the same symbol. Using the series expansions at www.emathhelp.net/calculators/calculus-1/taylor-and-maclaurin-series-calculator, it can be simply verified that $H^{(i)}(0) = 0$ for $1 \leq i \leq 16$ and

$$H^{(17)}(x) = \left[\frac{(-44)}{45 \cdot 10080} - \frac{17}{120 \cdot 1260} + \frac{17}{15 \cdot 12600} + \frac{11}{90 \cdot 1008} \right] \cdot x^{17} \sim$$

$$\sim 0.00000176366 \cdot x^{17}, \quad x \sim 0+;$$

moreover,

$$x^2 (1 - \cos x)^2 (3 \sin x - x \cos x - 2x)^2 \sim \frac{x^{16}}{1440}, \quad x \to 0+ .$$

This simply implies that $F'(0) = F''(0) = 0$ and $F'''(0) > 0$, so that the point $x = 0$ is a local minimum of the function $F(\cdot)$ and the function $F(\cdot)$ strictly increases in a right neighborhood of the point $x = 0$.

Due to (3), we have $(f_\alpha'(x))_{x=2n\pi} = 3(2n\pi)^{-1-\alpha} > 0$, $n \in \mathbb{N}$; moreover, due to (2) and (5), we have:

$$f_\alpha'(x) > 0 \Leftrightarrow F(x) < \alpha, \;\; x \in (0, \infty) \setminus 2\mathbb{N}\pi. \tag{6}$$

Further on, it can be simply proved that $F(2n\pi \pm 0) = -\infty$ for all $n \in \mathbb{N}$, as well as that

$$0 < F(2n\pi + (\pi/2)) = (2n\pi - 1 + (\pi/2)) \Big[(2n\pi + (\pi/2))(2n\pi - 2 + (\pi/2)) + (4n+1)\pi - 3 \Big]$$

tends to $+\infty$ as $n \to +\infty$. This immediately implies the following:

Proposition 1. *For each $\alpha > 0$, there exists $n_\alpha \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_\alpha$, there exists at least one local maximum of the function $f_\alpha(\cdot)$ on the interval $(2n\pi, 2n\pi + (\pi/2))$ and at least one local minimum of the function $f_\alpha(\cdot)$ on the interval $(2n\pi + (\pi/2), (2n+2)\pi)$.*

Furthermore, the foregoing arguments imply that, for every $n \in \mathbb{N} \setminus \{1\}$, there exist points $a_n \in ((2n-2)\pi, 2n\pi)$ and $\alpha_n \geq F(2n\pi + (\pi/2))$ such that $\lim_{n \to +\infty} \alpha_n = +\infty$ and $\alpha_n := F(a_n) = \max_{x \in ((2n-2)\pi, 2n\pi)} F(x)$; this also holds for $n = 1$, when $\alpha_1 \sim 1.982$ and $a_1 \sim 5.162$ (please, see https://www.desmos.com/calculator/73nvheiv5e). Using this numerical simulation, we may conclude that the following result holds good:

Proposition 2. *There exists a unique number $a_1 \sim 5.162$ such that the function $F(\cdot)$ is strictly increasing on $(0, a_1)$ and strictly decreasing on $(a_1, 2\pi)$. Furthermore, $\alpha_1 := F(a_1) \sim 1.982$ and the function $F(\cdot)$ has a unique real zero $\xi_1 \sim 5.763$ on the interval $(0, 2\pi)$.*

Using the above numerical simulation with www.desmos.com, it follows that the unique solution of the equation $F(x) = 4/3$ on the interval $(0, 2\pi)$ has the value $\zeta_1 \sim 5.611$. As an immediate consequence of Proposition 2 and the foregoing arguments, we have the following result which is deduced here with the help of computer software, without a pure analytical proof:

Theorem 4.

(i) *If $0 < \alpha \leq 4/3$, then $5.162 < x_\alpha < 5.763$, $\lim_{\alpha \to 4/3-} x_\alpha = \zeta_1$, $\lim_{\alpha \to 0+} x_\alpha = \xi_1$ and the mapping $x \mapsto x_\alpha$, $0 < \alpha \leq 4/3$ is continuous and strictly decreasing.*

(ii) *If $4/3 < \alpha \leq \alpha_1$, then $x_\alpha \in (0, x_1)$, $\lim_{\alpha \to 4/3+} x_\alpha = 0$, $\lim_{\alpha \to \alpha_1 -} x_\alpha = a_1$ and the mapping $x \mapsto x_\alpha$, $4/3 < \alpha \leq \alpha_1$ is continuous and strictly increasing.*

Remark 1. Suppose that $\alpha = 1$. Then x_1 is a unique real zero of the equation $F(x) = 1$ on $(0, 2\pi)$ and numerically we can calculate that $x_\alpha \sim 5.667$. Similarly, if $\alpha = 4/3$, then $x_{4/3}$ is a unique real zero of the equation $F(x) = 4/3$ on $(0, 2\pi)$ and we have $x_{4/3} \sim 5.611$.

It should be observed that the sequence (α_n) is not monotone but $\alpha_1 < \alpha_2 < \cdots < \alpha_{10}$; for more details, see https://www.desmos.com/calculator/l85rfiwvz8. We define the sequence (β_n) recursively as follows. Set $\beta_n := \alpha_n$ for $1 \leq n \leq 10$. If $\beta_1, \cdots, \beta_n$ are determined $(n \geq 10)$, then there exists an integer $s \in \mathbb{N}$ such that $\alpha_s > \beta_n$ (because $\lim_{s \to +\infty} \alpha_s = +\infty$) so that the set $B_n := \{s \in \mathbb{N} : \alpha_s > \beta_n\}$ is not empty. We set $\beta_{n+1} := \alpha_{\min(B_n)}$. Then (β_n) is a strictly increasing sequence of positive real numbers and $\lim_{n \to +\infty} \beta_n = +\infty$ since $\beta_n \geq \alpha_n$ for all $n \in \mathbb{N}$. Let $(\beta_n = \alpha_{k_n})$ for some strictly increasing sequence (k_n) of natural numbers, and let $n \in \mathbb{N}$ be fixed. Then it can be simply shown that there exists a unique number $\zeta_{n+1} \in ((2k_{n+1} - 2)\pi, a_{k_{n+1}})$ such that $F(\zeta_{n+1}) = \beta_n$ and $F(\zeta) < \beta_n$ for $\zeta \in ((2k_{n+1} - 2)\pi, \zeta_{n+1})$. Then clearly $\zeta_{n+1} > a_{k_n}$ for all $n \in \mathbb{N}$.

Using an elementary argumentation involving (6) and our construction of the sequence (β_n), we may conclude that the following result holds:

Theorem 5. *If there exists $n \in \mathbb{N}$ such that $\beta_n < \alpha \leq \beta_{n+1}$, then we have $x_\alpha \in ((2k_{n+1} - 2)\pi, 2k_{n+1}\pi)$, $\lim_{\alpha \to \beta_{n+1}-} x_\alpha = a_{k_{n+1}}$ and $\lim_{\alpha \to \beta_n +} x_\alpha = \zeta_{n+1}$. Furthermore, the mapping $\alpha \mapsto x_\alpha$, $\alpha > 4/3$, is strictly increasing, continuous on the set $\bigcup_{n \in \mathbb{N}} (\beta_n, \beta_{n+1})$ and has the discontinuities at the poins $\alpha = \beta_n$ $(n \in \mathbb{N})$ of the first kind.*

Remark 2. Suppose that $\alpha = 2$. Numerically we can prove that $a_2 \sim 12.07$, $\alpha_2 = \beta_2 \sim 8.596$, $x_\alpha \sim 9.548$ and $\zeta_\alpha \sim 9.6$.

Concerning the above consideration, which leans heavily on the use of numerical simulations, we would like to propose the following open problems:

1. Let $n \in \mathbb{N}$. Is it true that the point a_n is the only local extrema of the function $F(\cdot)$ on the interval $((2n - 2)\pi, 2n\pi)$?
2. Can we describe some properties of the sequence (k_n) in more detail?

We continue by exploring the monotony of the function $g_\alpha(\cdot)$ on the interval $(0, \pi)$, where $\alpha > 0$. Arguing as above, we may conclude that, for every $x \in (0, \pi)$, we have

$$g_\alpha'(x) = \frac{1}{3}\big[G(x) - \alpha\big]\big(\sin x - x\cos x\big)^{-1-\alpha}(3\sin x - x\cos x - 2x)\sin x,$$

with

$$G(x) := \frac{(\sin x - x\cos x)\Big[x(2\cos x + x\sin x - 2) - (3\sin x - x\cos x - 2x)\Big]}{x^2(3\sin x - x\cos x - 2x)\sin x}.$$

Using the graphic calculator https://www.desmos.com/calculator/afjhgutr8u, it can be easily seen that $G(0+) = 4/3$ as well as that $G(\cdot)$ is strictly monotonically increasing on the interval $(0, \pi)$. Keeping in mind (2), this simply implies that the function $g_\alpha(\cdot)$ is strictly monotonically decreasing on the interval $(0, \pi)$ if and only if $0 < \alpha \leq 4/3$. If $\alpha > 4/3$, then there exists a uniquely determined number $x_\alpha \in (0, \pi)$ such that the function $g_\alpha(\cdot)$ is strictly monotonically decreasing on the interval $(0, x_\alpha)$. Furthermore, x_α strictly increases with respect to the parameter $\alpha > 4/3$ and numerically we can prove that $x_2 \in (2.2, 2.3)$. We would like to ask whether we can give a pure analytical proof showing that the function $G(\cdot)$ is strictly monotonically increasing on the interval $(0, \pi)$?

For more details about the inequalities for mixed trigonometric polynomials (MTP polynomials), we refer the reader to the paper [7] by B. Malešević, M. Makragić and the doctoral dissertation [8] of B. Banjac.

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МОНОТОННОСТЬ НЕКОТОРЫХ КЛАССОВ ФУНКЦИЙ, СВЯЗАННЫХ С НЕРАВЕНСТВОМ КУСЫ — ГЮЙГЕНСА

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Теория аналитических неравенств по-прежнему является очень популярной областью исследований многих авторов. В этой заметке мы анализируем монотонность некоторых классов функций, связанных с неравенством Кусы — Гюйгенса, и предлагаем некоторые открытые проблемы в этом направлении.

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