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A REALIZATION THEOREM IN THE PROBLEM OF A STRICT ANALYTICAL CLASSIFICATION OF TYPICAL GERMS OF SEMIHYPERBOLIC MAPPINGS

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We consider the problem of the analytic classification of germs of semi-hyperbolic mappings by the example of germs of simplest form. The final theorem, which is necessary for constructing an analytical classification, is proved — a theorem on the realization of elements of a certain functional space as functional modules of the constructed classification. To prove the theorem, the method of almost complex structures is used.

Keywords: *semihyperbolic mapping, analytic classification, functional module, realization.*

Introduction

In this paper, we consider the well-known problem of reducing a germ of a holomorphic map by an analytic change of coordinates to its simplest form — its normal form. As a normal form of the analytical classification, it is customary to choose a formal normal form. This problem has been solved for Poincare-type germs and non-resonant Siegel-type germs; a survey on this issue can be found in [1]. The first significant results in the study of the analytical normalization of germs of resonant mappings of Siegel type were obtained in the 1930s by Birkhoff [2] and others. It turned out that, in contrast to the previously obtained results, in this case the analytical normalization of the sprout is possible only on some sectorial areas that form a cover of the punctured neighbourhood of the origin. However, only in the early 1980s, in the independent works of Voronin [3] and Ecalle [4], the problem of analytical classification was completely solved for germs of one-dimensional holomorphisms (parabolic mappings).

Thus, the algorithm for constructing the analytical classification of germs of resonant Siegel type holomorphisms was developed as follows [5]. The neighbourhood of the origin is covered by a set of sectorial areas. On each of them, a holomorphic coordinate change is constructed that normalizes the germ. At the intersection of sectorial regions, a pair of normalizing mappings naturally arises. The transition functions of the normalizing atlas deliver the desired functional invariants. Functional invariants are elements of some naturally occurring functional space. At the final stage, the realization of all elements of this functional space as the invariants of the constructed classification is proved.

As mentioned above, this program was implemented completely for germs of parabolic mappings in dimension 1; quite a bit was done for large dimensions. To date, for one simplest class of germs of two-dimensional mappings (formally equivalent to the germ F_λ of the map $F_\lambda = \left(\frac{x}{1-x}, e^\lambda y\right)$): normalizing sectorial mappings [6] are constructed and the space of functional invariants [7] is described. This work completes the construction of a classification of germs of the simplest form. Note that the arguments presented to

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prove the implementation theorem in the simplest case can be easily generalized to the class of typical germs of semihyperbolic mappings described in [8].

Particular results in the study of germs of Siegel-type resonance mappings (including semihyperbolic mappings on the plane) were obtained by Stolovich [9] and Ueda [10; 11]. However, a normalizing atlas was not built in these works.

1. Preliminary information and the main result

The germ F of a holomorphic resonance map $F:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ will be called *semihyperbolic* if one of its the multipliers are parabolic (equal to unity), and the other is hyperbolic (not modulo zero or unity). A semi-hyperbolic map will be called *typical* if in the expansion $F(x,y)=\left(x+cx^2+...,e^{\lambda}y+...\right)$ the constant «c» is not equal to zero. Thus, the germ F_λ of the map

$$F_\lambda=\left(\frac{x}{1-x},e^{\lambda}y\right),\ \lambda\in\mathbb{R}_+,$$

is typical semihyperbolic.

As usual, two mappings F and $\tilde{F}$ with different domains of definition U and $\tilde{U}$ will be called *analytically equivalent* if there exists a holomorphic change of coordinates $H:\tilde{U}\to U$ conjugating $\tilde{F}$ with F:

$$F\circ H=H\circ\tilde{F}.$$

Equivalence will be called *strict* if the conjugating change of coordinates H has the form

$$H(x,y)=\left(x+o(x^2),y+o\left(x\right)\right),\ x\to 0. \tag{1}$$

Replacements of the coordinates of form (1) will be called *normalized*. Two germs will be called *strictly analytically equivalent* if their strictly analytically equivalent representatives exist.

Two germs F and $\tilde{F}$ will be called *strictly formally equivalent*, if there exists a formal normalized change of coordinates H for which (1) is true as the equality of formal rows. Denote by $\mathbf{F}_\lambda$ the class of holomorphic germs strictly formally equivalent to the germ F_λ. The germ F_λ (as well as its representative F_λ) will be called *normal form* of the class $\mathbf{F}_\lambda$.

Suppose that there exists some covering of a cut-off neighbourhood of the origin $\{0<|x|<\tilde{\varepsilon}\}\times\{|y|<\varepsilon\}$ of the form $\{\Omega\}=\cup\Omega_j$.

Remark 1. Throughout what follows, we will consider $j\in\mathbb{Z}_4$, unless otherwise specified.

The paths of the coverage area intersect, so for all j $\Omega_{j,j+1}=\Omega_j\cap\Omega_{j+1}\neq 0$ is satisfied (there are no other intersections). Let a normalized sectorial normalizing mapping H_j be constructed on each Ω_j (for the construction of such normalizing mappings for germs of the class $\mathbf{F}_\lambda$, see [6] and the link to the archive). Then at the intersection $\Omega_{j,j+1}$ a pair of normalizing ones is defined: H_j and H_{j+1}. The mapping

$$\Phi_{j,j+1}\stackrel{\text{def}}{=}H_{j+1}^{-1}\circ H_j$$

with the domain $\Omega_j\cap\left(H_j^{-1}\left(H_{j+1}\left(\Omega_{j+1}\right)\right)\right)$ will be called a *transition function*.

In [7], transition functions of germs of the class $\mathbf{F}_\lambda$ were constructed. In coordinates (which are selected as the first integrals of the normal form F_λ), we get

$$(t,\tau) = I(x,y) \stackrel{\text{def}}{=} \left(t = e^{-\frac{2\pi i}{x}}, \tau = ye^{-\frac{\lambda}{x}}\right), \tag{2}$$

transition functions are as follows:

$$\Upsilon_{j,j+1}(t,\tau) \stackrel{def}{=} I \circ \Phi_{j,j+1} \circ I^{-1} = \left(t\left(1 + A_{j,j+1}\right), (\tau + B_{j,j+1})(1 + A_{j,j+1})^{-\frac{\lambda}{2\pi i}}\right),$$

where $A_{j,j+1}$ и $B_{j,j+1}$ satisfy the following conditions:

$$\left.\begin{array}{l}
A_{1,2} = 0,\ B_{1,2} = C \in \mathbb{C};\\
A_{4,1} = A_{4,1}(t,\tau),\ B_{4,1} = B_{4,1}(t,\tau)\ \text{ holomorphic in } \Theta_{4,1} = (\mathbb{C}^2, 0),\\
\text{and } A_{4,1}(t,\tau) = O(t^2),\ B_{4,1}(t,\tau) = O(t),\ t \to 0;\\
A_{2,3} = A_{2,3}(t,\tau),\ B_{2,3} = B_{2,3}(t,\tau)\ \text{ holomorphic in } \Theta_{2,3} = (\mathbb{C}, \infty) \times (\mathbb{C}, 0),\\
\text{and } A_{2,3}(t,\tau) = O(1),\ B_{2,3}(t,\tau) = O(t^{-1}),\ t \to \infty;\\
A_{3,4} = A_{3,4}(\tau),\ B_{3,4} = B_{3,4}(\tau)\ \text{ holomorphic in } \Theta_{3,4} = (\mathbb{C}, 0),\\
\text{and } A_{3,4}(\tau) = O(\tau),\ B_{3,4}(\tau) = O(\tau^2),\ \tau \to 0.
\end{array}\right\} \tag{3}$$

Definition 1. The built set $(A_{j,j+1}, B_{j,j+1}, C)$ we will call the *functional module* of the germ $\mathsf{F} \in \mathbf{F}_\lambda$ and denote by m_F.

Denote by $\mathbf{M}_\lambda$ the functional space of all kinds of sets $m = (A_{j,j+1}, B_{j,j+1}, C)$, where $A_{j,j+1}$, $B_{j,j+1}$ и C satisfy (3). In [7], it was shown that for any $\mathsf{F} \in \mathbf{F}_\lambda$, there is the set $m_F \in \mathbf{M}_\lambda$. In this paper, the following converse statement will be proved:

Theorem 1. [Theorem on realization]. *For any set $m \in \mathbf{M}_\lambda$ there is a germ $\mathsf{F} \in \mathbf{F}_\lambda$, for which the set m is the functional module $m = m_F$.*

2. Proof of the theorem

2.1. Definition of a sectorial coverage

Choose parameters $\delta, \tilde{\delta} \in \left(0, \frac{\pi}{2}\right)$, $R, \varepsilon \in \mathbb{R}$ as $0 < \delta < \tilde{\delta} < \pi/2$, $R \ll 1$, $0 < \varepsilon \gg 1$.

Definition 2. A left sectorial domain S^l on a ξ-plane is an addition to the convex hull of a disk union $B_R \stackrel{def}{=} \{|\xi| \leq R\}$, a domain «on the right» of the straight line $L_{R+3} \stackrel{\text{def}}{=} \{\xi \in \mathbb{C} : Re\xi = R + 3\}$ (for any ξ from a domain «on the right», $Re\xi > R + 3$ holds) and a sector $\{\xi \in \mathbb{C} : |\arg\xi| \leq \tilde{\delta}\}$.

Definition 3. An upper left S_1 (left S_2) sectorial domain on a ξ-plane is an intersection of S^l and a half-plane $\{\xi \in \mathbb{C} : \delta < \arg\xi < \pi + \delta\}$ ($\{\xi \in \mathbb{C} : \pi - \delta < \arg\xi < 2\pi - \delta\}$).

Definition 4. An upper right S_4 (lower right S_3) sectorial domain on a ξ-plane is an intersection of a domain «on the right» of the straight line $L_R \stackrel{def}{=} \{\xi \in \mathbb{C} : Re\xi = R\}$ and a half-plane $\{\xi \in \mathbb{C} : -\delta < \arg\xi < \pi - \delta\}$ ($\{\xi \in \mathbb{C} : \pi + \delta < \arg\xi < 2\pi + \delta\}$).

Definition 5. A sectorial domain X_j on an x-plane is a prototype of a domain S_j when mapping $\varkappa : x \mapsto \xi = -\frac{1}{x}$:

$$X_j \stackrel{\text{def}}{=} \varkappa^{-1}(S_j).$$

Choosing the appropriate parameters of sectorial domains, without loss of generality, we can assume that a set $\{\Omega\}$ of sectorial areas $\Omega_j = X_j \times \{|y| < \varepsilon\}$ forms a covering of the cut-out neighbourhood of the origin $\{0 < |x| < \tilde{\varepsilon}\} \times \{|y| < \varepsilon\}$, where $\tilde{\varepsilon} = 1/R$. We will call this covering *standard*.

From the definition of the domains of standard coverage it follows that Ω_j and Ω_{j+1} intersect. The intersection of the areas of standard coverage is called a *standard intersection* and denoted by $\Omega_{j,j+1}$. The set of standard intersections, respectively, is denoted by $\{\Omega_{j,j+1}\}$.

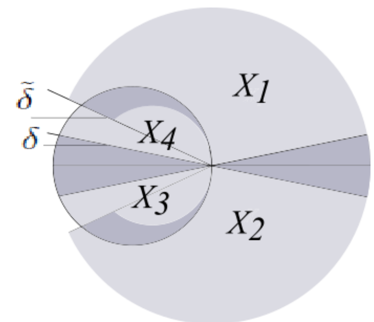

Fig. 1. Set of areas X_j of standard coverage

Remark 2. Note that by standard intersection the standard coverage can be *uniquely* restored.

Definition 6. We say that the set of domains $A = \{A_j\}$ is inscribed in the set of domains $B = \{B_j\}$, if for any j, a domain A_j is a subdomain of B_j.

Definition 7. A set of domains $\{E_j\}$ (a set of $\{E_{j,j+1}\}$) is called a *sectorial covering* (*sectorial intersection*, respectively), if a standard covering is inscribed in $\{E_j\}$ and $\{E_j\}$ is inscribed in a standard covering (for $\{E_{j,j+1}\}$, it is analogously).

2.2. Construction of a topological space $\mathcal{M}$

Let $m = (A_{j,j+1}, B_{j,j+1}, C)$ be an element of a space $\mathbf{M}_\lambda$. Define a set of mappings $\{\Upsilon_{j,j+1}(t,\tau)\}$ with domains $\Theta_{1,2} = \mathbb{C}^2$, $\Theta_{2,3} = (\mathbb{C}, \infty) \times (\mathbb{C}, 0)$; $\Theta_{3,4} = \mathbb{C} \times (\mathbb{C}, 0)$, $\Theta_{4,1} = (\mathbb{C}^2, 0)$ with m by the following rule

$$\Upsilon_{1,2}(t,\tau) = (t, \tau + C),$$
$$\Upsilon_{j,j+1}(t,\tau) = \Big(t(1 + A_{j,j+1}(t,\tau)), (\tau + B_{j,j+1}(t,\tau))(1 + A_{j,j+1}(t,\tau))^{-\frac{\lambda}{2\pi i}}\Big), \ j = 2, 4,$$
$$\Upsilon_{3,4}(t,\tau) = \Big(t(1 + A_{3,4}(\tau)), (\tau + B_{3,4}(\tau))(1 + A_{3,4}(\tau))^{-\frac{\lambda}{2\pi i}}\Big).$$

Domains $\Theta_{j,j+1}$ we call *natural* domains of functions $\Upsilon_{j,j+1}$. Write $\Upsilon_{j,j+1}$ in the coordinates (x, y), where $(t,\tau) = I(x, y)$ are from (2),

$$\Phi_{j,j+1} \overset{\text{def}}{=} I^{-1} \circ \Upsilon_{j,j+1} \circ I.$$

A part of a preimage of the domain $\Theta_{j,j+1}$ when mapping I, where both components of the mapping $\Phi_{j,j+1}$ are holomorphic, is a *natural* domain of the mapping $\Phi_{j,j+1}$. Natural domains of definition will be denoted by $\operatorname{dom}\Phi_{j,j+1}$. Note that by the definition of components $A_{j,j+1}$ and $B_{j,j+1}$

$$\forall N \in \mathbb{N}: \ \Phi_{j,j+1}(x, y) - (x, y) = (o(|x|^{-N}), o(|x|^{-N})), \ x \to 0, \ (x, y) \in \operatorname{dom}\Phi_{j,j+1} \quad (4)$$

holds. Without loss of generality, we can assume that, by virtue of (4), there is a standard intersection $\{U_{j,j+1}\}$ inscribed in a set $\{\operatorname{dom}\Phi_{j,j+1}\}$ such that

— sets $\{\Phi_{j,j+1}(U_{j,j+1})\}$ and $\{\Phi^{-1}_{j,j+1}(U_{j,j+1})\}$ are also inscribed in $\{\operatorname{dom}\Phi_{j,j+1}\}$;

— some standard intersection may be inscribed in a set $\{U_{j,j+1} \cap \Phi_{j,j+1}(U_{j,j+1}) \cap \Phi^{-1}_{j,j+1}(U_{j,j+1})\}$.

Thereby sets

$$\{E_{j,j+1}\} \overset{\text{def}}{=} \{\Phi^{-1}_{j,j+1}(U_{j,j+1}) \cap U_{j,j+1}\}, \quad \{Z_{j,j+1}\} \overset{\text{def}}{=} \{U_{j,j+1} \cap \Phi_{j,j+1}(U_{j,j+1})\}$$

are sets of sectorial intersections. Moreover

$$\Phi_{j,j+1}(E_{j,j+1}) = Z_{j,j+1}.$$

We choose some sectorial domain V_j so that it «connects» the domains $E_{j,j+1}$ and $Z_{j-1,j}$ to a simply connected sectorial domain. In addition, the conditions of Remarks 3 and 5 are also be fulfilled (see below). The possibility of constructing this domain follows from the construction of domains of a standard intersection. Build a set of sectorial domains $\{\Omega_j\}$ by the rule

$$\Omega_j \stackrel{\text{def}}{=} Z_{j-1,j} \cup V_j \cup E_{j,j+1}. \tag{5}$$

Consider the topological space $\mathcal{M}$ are obtained from $\{\Omega_j\}$ after gluing by mappings $\Phi_{j,j+1}$ according to the following rule. Let the elements of a space $\Omega_j \times \{j\}$ be points $(x, y) \times \{j\}$. Then for every j gluing mappings $\Phi_{j,j+1}$ act from $E_j \times \{j\}$ to the set $Z_{j+1,j+2} \times \{j+1\}$. The elements of $\mathcal{M}$ are

$$\begin{array}{l} \text{points } (x,y)\times\{j\}, \ \ \text{if } (x,y)\in\Omega_j\backslash(Z_{j,j+1}\cup E_{j,j+1}), \\ \text{pairs } ((x,y)\times\{j\},\Phi_{j,j+1}(x,y)\times\{j+1\}), \ \ \text{if } (x,y)\in E_{j,j+1}. \end{array}$$

Let $\pi_j : \Omega_j \to \mathcal{M}$ be natural injections, $W_j \stackrel{\text{def}}{=} \pi_j(\Omega_j)$. The topology on $\mathcal{M}$ is defined in the standard way: a set $U \subset \mathcal{M}$ is open, if and only if for all j sets $\pi_j^{-1}(U \cap W_j)$ are open. Then (W_j, π_j^{-1}) are natural maps of a manifold $\mathcal{M}$, and $\Phi_{j,j+1}$ are transition functions: $\pi_{j+1}^{-1} \circ \pi_j = \Phi_{j,j+1}$.

Remark 3. Without loss of generality, we can assume that, due to the choice of areas V_j, the space $\mathcal{M}$ is a *Hausdorff space*, therefore, maps (W_j, π_j^{-1}) define the structure of a complex manifold on $\mathcal{M}$. A collection $W = \{W_j\}$ forms an open covering of the manifold $\mathcal{M}$.

Introduce an auxiliary function

$$e(x,p,q) = \exp\left(\frac{-\exp\left(\frac{-1}{(q-x)^2}\right)}{(p-x)^2}\right), \quad x \in (p,q),$$

its graph is presented on Fig. 2.

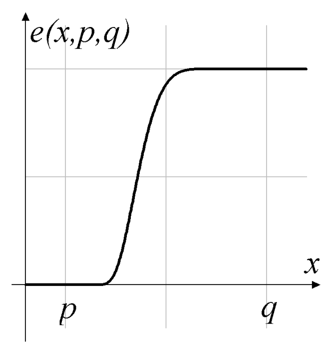

Fig. 2.

Remark 4. A function $e(x,p,q)$ is infinitely smooth and bounded on the closure of the domain.

Remark 5. By the construction of the domains $E_{j,j+1}$, $Z_{j,j+1}$ and the choice of V_j, there is a standard intersection $\{\omega_{j,j+1}\}$ with parameters d, $\tilde{d}$ and $\tilde{R}$ such that the domains $\omega_{j,j+1}$ are inscribed in $E_j \cap Z_j$ (for suitable $0 < d < \delta$, $\tilde{\delta} < \tilde{d} < \pi/2$, $\tilde{R} > R$, where δ, $\tilde{\delta}$ and R are parameters of the standard intersection $\{U_{j,j+1}\}$).

Let curves $\arg x = \alpha_{j,j+1}(|x|)$ and $\arg(x) = \beta_{j,j+1}(|x|)$ define the «left» and «right» borders of the sectorial area $\omega_{j,j+1}$ (i. e. $\omega_{j,j+1} = \{(x,y) : \alpha_{j,j+1}(|x|) \le \arg(x) \le \beta_{j,j+1}(|x|)\}$).

Using a standard intersection $\{\omega_{j,j+1}\}$, we construct a standard covering $\{\omega_{j,j+1}\}$. Define on each ω_j a *cutoff function* θ_j by the rule

$$\theta_j = \begin{cases} 1, \ (x,y) \in \omega_j \backslash (\omega_{j,j+1} \cup \omega_{j-1,j}), \\ 1 - e(\arg x, \alpha_{j-1,j}(|x|), \beta_{j-1,j}(|x|)), \ (x,y) \in \omega_{j-1,j}, \\ e(\arg x, \alpha_{j,j+1}(|x|), \beta_{j,j+1}(|x|)), \ (x,y) \in \omega_{j,j+1}. \end{cases}$$

Note that the functions θ_j are continued by the same formulas with ω_j on Ω_j. Without loss of generality, we can assume that θ_j are defined on Ω_j.

2.3. An almost complex structure

Define a function $\theta_j \circ \pi_j^{-1}$ on W_j and continue it on the whole $\mathcal{M}$ with W_j, setting it equal to zero outside W_j. Let

$$G \stackrel{\text{def}}{=} \sum_{j=1}^{4} \theta_j \circ \pi_j^{-1} \cdot \pi_j^{-1}.$$

Then G acts from $\mathcal{M}$ to $\mathbb{C}^2$. Denote an image of the manifold $\mathcal{M}$ when mapping G by $\mathcal{N}_0$. By the construction of the manifold $\mathcal{M}$ and the mapping G it follows that $\mathcal{N}_0 = (\mathbb{C}_*, 0) \times (\mathbb{C}, 0)$. Denote a restriction of G onto W_j by G_j.

Remark 6. It is easy to verify that if the radii of the initial regions are sufficiently small, then G is injective on $\mathcal{M}$. Therefore, G_j is injective on W_j.

From Remark 6, it follows that there are inverse mappings $G^{-1} : \mathcal{N}_0 \to \mathcal{M}$ and $G_j^{-1} : G(W_j) \to W_j$.

Remark 7. The diffeomorphism $G : W \to \mathcal{N}_0$ sets on $\mathcal{N}_0$ an almost complex structure (ACS) on $\mathcal{N}_0$ [12] induced by a complex structure of $\mathcal{M}$ (Remark 3).

Remark 8. By (4) and a restriction of the cutoff functions (Remark 4) it follows that

$$\forall N \in \mathbb{N}: \; G_j \circ \pi_j - (x, y) = \left(o(x^N), o\left(x^N\right)\right), \; x \to 0, \; (x, y) \in \Omega_j,$$

$$\forall N \in \mathbb{N}: \; \pi_j^{-1} \circ G_j^{-1} - (x, y) = \left(o(x^N), o\left(x^N\right)\right), \; x \to 0, \; (x, y) \in G(W_j). \qquad (6)$$

Moreover, Remark 4 and the type of the sectorial coverage $\{\Omega_j\}$ implies the possibility of the terminative differentiation of the expression (6).

Denote $\pi_j^{-1} \circ G_j^{-1} = (a_j, b_j)$, $\omega_1^j = d(a_j)$, $\omega_2^j = d(b_j)$. Define on $\mathcal{N}_0$ two 1-forms according to the rules

$$\omega_1 = \sum_{j=1}^{4} \theta_j \circ \pi_j^{-1} \circ G_j^{-1} \cdot \omega_1^j, \qquad \omega_2 = \sum_{j=1}^{4} \theta_j \circ \pi_j^{-1} \circ G_j^{-1} \cdot \omega_2^j.$$

Note that by construction, functions $\theta_j \circ \pi_j^{-1} \circ G_j^{-1}$ are nonzero only on $G(W_j)$. Thus, the 1-forms ω_1 and ω_2 are actually the sum of two terms (at the intersection of $G(W_j)$ and $G(W_{j+1})$) and exactly one term outside the intersections.

Let Ω_h be a space of holomorphic 1-forms on $\mathcal{N}_0$ in terms of ACS, by Remark 7.

Lemma 1. *The radii of the source regions can be chosen small enough so that* $\Omega_h = \langle \omega_1, \omega_2 \rangle$.

Доказательство. By definition of induced ACS, the space $\Omega_h(B)$ of holomorphic ACS 1-forms is spanned by the differentials of the components of the mapping $G_j^{-1}(B)$ at the points $B = G(A)$ such that $A \in W_j$: $\Omega_h(B) = \langle \omega_1^j, \omega_2^j \rangle$. By the definition of 1-forms ω_1^j, ω_2^j, since $(a_{j+1}, b_{j+1}) = \Phi_{j,j+1}(a_j, b_j)$, then $\langle \omega_1^{j+1}, \omega_2^{j+1} \rangle = \langle \omega_1^j, \omega_2^j \rangle$ (where they are all defined — i.e. at the intersection of sectors $G(W_j)$ and $G(W_{j+1})$). Since forms ω_1 and ω_2 are linear combinations of the forms ω_1^j and ω_2^j, then $\langle \omega_1^j, \omega_2^j \rangle \subset \Omega_h$. Taking into account the asymptotic behavior (6) and the definition of cutoff functions, we obtain the linear independence of the 1-forms ω_1 and ω_2. Here the required follows: for every $B \in \mathcal{N}_0$ $\Omega_h(B) = \langle \omega_1, \omega_2 \rangle$. □

Thus, there is set an almost complex structure (ACS) on $\mathcal{N}_0$, that is induced by the diffeomorphism G.

2.4. The end of the proof of the realization theorem

The almost complex structure defined in this way on $\mathcal{N}_0$ is integrable (or, according to the terminology [12], resolvable), since it is induced.

Extend ACS by continuity with $\mathcal{N}_0 = (\mathbb{C}_*, 0) \times (\mathbb{C}, 0)$ on $\mathrm{Int}(\overline{\mathcal{N}_0}) = \mathcal{N} = (\mathbb{C}^2, 0)$, supposing

$$\Omega_h(0, y) = \langle dx, dy \rangle,\ (0, y) \in \mathcal{N} \backslash \mathcal{N}_0. \tag{7}$$

By Remark 8 and Lemma 1, it follows that the defined structure on $\mathcal{N}$ is smooth, that is ACS on $\mathcal{N}$. Moreover, since the original ACS was integrable on $\mathcal{N}_0$, then the ACS extended on $\mathcal{N}$.

Theorem 2. [12, Newlander — Nirenberg, Theorem 5.7.4]. *Every resolvable ACS is defined (locally) by a single analytical structure.*

From the Newlander — Nirenberg theorem it follows that there exists an injective mapping $P : \tilde{\mathcal{N}} \to (\mathbb{C}^2, 0)$, that is defined in some neighbourhood $\tilde{\mathcal{N}} \subset \mathcal{N}$ of a point $(0, 0)$ and holomorphic in terms of ACS (i.e. it is such that the linear subspace of differential 1-forms Ω_h is generated by the differentials of the components of the mapping P in the points of $\tilde{\mathcal{N}}$). Moreover, $P \in C^\infty(\tilde{\mathcal{N}})$. Reducing, if necessary, the radii of the initial regions, without loss of generality, we can assume that $\tilde{\mathcal{N}} = \mathcal{N}$.

Remark 9. If the mapping P satisfies the Newlander — Nirenberg theorem, then, for any holomorphic and injective in some sufficiently small neighbourhood $(\mathbb{C}^2, 0)$ mapping T, a composition $T \circ P$ also satisfies the theorem.

Let $J_j = P \circ G_j$. Then the mapping J_j acts from $W_j \subset \mathcal{M}$ to $\mathbb{C}^2$ and translates a complex structure $\mathcal{M}$ into a complex structure $\mathbb{C}^2$. Therefore, J_j is holomorphic. A mapping $J = P \circ G$ is a biholomorphism of the manifold $\mathcal{M}$ on $P(\mathcal{N}_0)$.

Since mappings of the gluing $\Phi_{j,j+1}$ commute with a normal form F_λ, then on some subdomain $\tilde{\mathcal{M}} \subset \mathcal{M}$ the holomorphic mapping $\tilde{F}$ is correctly defined and coincides with F_λ in natural maps.

The biholomorphism J conjugates $\tilde{F}$ with some holomorphic mapping $\overset{\circ}{F}$, which, in turn, is defined on a subdomain $\mathcal{N}_1 = P(G(\tilde{\mathcal{M}}))$. As usual, choosing the parameters of the initial regions small enough, without loss of generality we can assume that $\tilde{\mathcal{M}} = \mathcal{M}$, $\mathcal{N}_0 = G(\mathcal{M})$, $\mathcal{N}_1 = P(\mathcal{N}_0)$.

Denote $\mathcal{N}_2 = \mathrm{Int}(\overline{N}_1)$. A part of the line $\{x = 0\}$ intersecting with $\mathcal{N}$ is denoted by L. Then $L = \mathcal{N} \backslash \mathcal{N}_0$. Then $\overset{\circ}{F}$ is holomorphic everywhere on $\mathcal{N}_2$ except $P(L)$. Note that the restriction P onto L is holomorphic (it follows from (7)), so that $P(L)$ is an analytical curve, and according to [13] it is a «thin set». Thus, according to the Riemann theorem (see [13, Theorem 2, ch. 3, §7]), $\overset{\circ}{F}$ continues to some $\dot{F}$, that is holomorphic on $\mathcal{N}_2 = (\mathbb{C}^2, 0)$.

Note that the mapping $\hat{G}_j \overset{\text{def}}{=} G_j \circ \pi_j$ is smooth (by construction) and conjugates the map F_λ with some smooth mapping $\hat{F}_j$ defined on $\hat{\Omega}_j = \hat{G}_j(\Omega_j) \subset \mathcal{N}$.

By construction, there is some sectorial area $\tilde{V}_j \subset V_j$ (V_j is from (5)) such that the cutoff θ_j equals to 1 on $\tilde{V}_j$. Then on this sectorial region, the space of holomorphic 1-forms Ω_h is generated by $\langle dx, dy \rangle$. That is, $\hat{G}_j$ is holomorphic on $\tilde{V}_j$. Consequently, the map P is holomorphic in the image $\hat{G}_j(\tilde{V}_j)$. Namely, the Taylor series mapping P centered at points in the domain $\hat{G}_j(\tilde{V}_j)$ (understood in the real sense) does not contain variables $\bar{x}, \bar{y}$. Since $(0, 0) \subset \partial \tilde{V}_j$, then from the asymptotic formulas of Remark 8, it follows that $(0, 0) \in \partial \hat{G}_j(\tilde{V}_j)$. Then, by continuity, the Taylor series of the mapping

P centered at the origin also does not contain $\bar{x}$ and $\bar{y}$. It follows that F_λ is formally equivalent to $\dot{F}$ (in the complex sense). That is to say, $\dot{F}$ is a semihyperbolic mapping.

According to the semi-formal classification of the theorem proved in [8], as well as Corollary 1 from it, for any positive integer N there is a holomorphic change of coordinates T_N, conjugating F_λ with a semi-hyperbolic mapping $\dot{F}$ up to a residual $\big(o(x^N), o(x^N)\big)$ when $x \to 0$. Then a mapping $H_j \overset{\text{def}}{=} T_2 \circ J_j \circ \pi_j$ defined on Ω_j, is holomorphic, normalized and conjugates the normal form F_λ with a holomorphic mapping F of the form $F(x,y) = F_\lambda(x,y) + (o\,(x^2)\,, o\,(x^2))$, $x \to 0$. So that F is strictly formally equivalent to F_λ, i.e. it is a representative of a germ of the class $\mathbf{F}_\lambda$.

From the construction of mappings H_j it follows that they are sectorial normalizing for F. Then the corresponding transition functions are equal to $\Phi_{j,j+1}$:

$$H_{j+1}^{-1} \circ H_j = \pi_{j+1}^{-1} \circ \pi_j = \Phi_{j,j+1}.$$

So, in accordance with the construction of modules of strict analytic classification, the initial set m is the module of the germ F: $m = m_F$. Thus, the realization theorem is proved.

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ТЕОРЕМА О РЕАЛИЗАЦИИ В ЗАДАЧЕ О СТРОГОЙ АНАЛИТИЧЕСКОЙ КЛАССИФИКАЦИИ ТИПИЧНЫХ РОСТКОВ ПОЛУГИПЕРБОЛИЧЕСКИХ ОТОБРАЖЕНИЙ

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Рассматривается задача об аналитической классификации ростков полугиперболических отображений на примере ростков простейшего вида. Доказана заключительная теорема, необходимая для построения аналитической классификации, — теорема о реализации элементов некоторого функционального пространства в качестве функциональных модулей построенной классификации. Для доказательства теоремы применяется метод почти комплексных структур.

Ключевые слова: *полугиперболическое отображение, аналитическая классификация, функциональный модуль, реализация.*

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