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ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR A MIXED PARABOLIC-HYPERBOLIC EQUATION

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We formulate and study a nonlocal problem with a special matching conditions and shift conditions for a parabolic-hyperbolic equation. Using the theory of integral equations we established the unique solvability of the considered problem.

Keywords: *nonlocal problem, parabolic-hyperbolic equation, method of integral equations.*

1. Introduction

The interest that arose in the first quarter of the last century in the theory of equations of mixed type has not diminished to this day. On the contrary, it is intensively developing in various directions, both in terms of the equations themselves and the range of problems considered for them, as well as the domain of consideration. While initially the focus was on problems for equations of mixed elliptic-hyperbolic type, since the 1960s, research has expanded to include equations of mixed elliptic-parabolic and parabolic-hyperbolic types.

The necessity of considering conjugation problems, where a parabolic equation is defined on one part of the domain and a hyperbolic equation on another, was first pointed out by I.M. Gelfand [1]. He discusses an example related to the motion of gas in a channel surrounded by a porous medium: in this case, the gas motion in the channel is described by the wave equation, while outside the channel, it is described by the diffusion equation. Y.A. Ufliand, in his article [2], describes the problem of the propagation of electric oscillations in composite lines, where on the section $0 < x < l$ of the semi-infinite line, losses are neglected, and the rest of the line is considered as a cable without leakage. As a result, he arrives at a problem for a parabolic-hyperbolic equation in the first quadrant of the plane.

Further research has also shown that many mathematical models of heat and mass transfer in capillary-porous media [3], electromagnetic field propagation in heterogeneous media [4], temperature field formation [5], and the motion of viscoelastic and viscous fluids [6] reduce to boundary value problems for equations of parabolic-hyperbolic type.

The first works dedicated to the study of boundary and initial-boundary problems for parabolo-hyperbolic equations are [7–10]. However, recently, researchers have increasingly focused on the formulation and study of non-local boundary problems with conditions similar to Frankl's conditions [11], Bitsadze–Samarski's conditions [12], shift conditions [13; 14], and integral conditions [15; 16]. For more information about non-local problems for equations of both integer and fractional order, we refer the reader to the references [17–20] and the references therein.

The relevance of studying such problems can be justified both by the internal need for theoretical generalizations of classical boundary problems for partial differential equations and by their practical applications. Moreover, the naturalness of formulating

problems where the boundary conditions represent relationships between the values of the unknown function computed at different points on the boundary was noted as early as 1896 by V.A. Steklov. Such boundary conditions also arise in mathematical modeling of problems in gas dynamics, plasma theory, laser radiation, and cell proliferation processes [21].

2. Preliminaries

In this section, we give the definitions and some properties of the some operators which will be used the throughout of the paper.

Let $\Gamma(z)$ be the Euler gamma function, and $\omega = \text{const} \neq -1, -2, \ldots$ Then the following series

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\omega}}{k!\ \Gamma(k+\omega+1)}$$

converges for every $z \in \mathbb{R}$. Typically, the sum of this series is denoted by $J_\omega(z)$ and is called the Bessel function of the first kind of order ω.

The function $J_\omega(z)$ has the following integral representation [22]:

$$J_\omega(z) = \frac{(z/2)^\omega}{\sqrt{\pi}\Gamma(\omega+1/2)} \int_{-1}^{1} (1-\xi^2)^{\omega-(1/2)} \cos(\xi z)\, d\xi, \quad \operatorname{Re}\omega > -1/2. \tag{1}$$

The function defined by the equality $\bar{J}_\omega(z) = \Gamma(\omega+1)(z/2)^{-\omega} J_\omega(z)$ is called the Bessel–Clifford function. It is evident that $\bar{J}_\omega(z)$ is an analytic function and $\bar{J}_\omega(0) = 1$.

Let a, b, m be be some real numbers, and λ be, in general, a complex number, where $a < b < +\infty$ and $m \in [a, b]$. Consider the following operators, introduced and studied in [23]:

$$A_{mx}^{n,\lambda}\left[f(x)\right] \equiv f(x) - \int_m^x f(t) \left(\frac{t-m}{x-m}\right)^n \frac{\partial}{\partial t} J_0[\lambda\sqrt{(x-m)(x-t)}\,]dt, \tag{2}$$

$$B_{mx}^{n,\lambda}[f(x)] \equiv f(x) + \int_m^x f(t) \left(\frac{x-m}{t-m}\right)^{1-n} \frac{\partial}{\partial x} J_0[\lambda\sqrt{(t-m)(t-x)}\,]dt, \tag{3}$$

$$C_{mx}^{0,\lambda}\left[g(x)\right] \equiv \operatorname{sign}(x-m) \left\{ \frac{d}{dx} g(x) + \frac{1}{2}\lambda^2 \int_m^x g(t) \bar{J}_1\left[\lambda(x-t)\right] dt \right\}, \tag{4}$$

where $n = 0, 1$, $f(x)$ and $g(x)$ are given functions, $f(x) \in C\left(a,b\right) \cap L\left(a,b\right)$ and $g(x) \in C^1\left(a,b\right) \cap L\left(a,b\right)$, with $L(a,b)$ denoting the space of integrable functions on (a,b).

In the work [23], the validity of the following theorems was proved:

Theorem 1. [23]. *If $f(x) \in C(a,b) \cap L(a,b)$, then for all $m \in [a,b]$ and $x \in (a,b)$, the following equalities hold:*

$$A_{mx}^{n,\lambda}\left\{B_{mx}^{n,\lambda}\left[f(x)\right]\right\} = f(x), \quad B_{mx}^{n,\lambda}\left\{A_{mx}^{n,\lambda}\left[f(x)\right]\right\} = f(x). \tag{5}$$

Theorem 2. [23]. *If $f(x) \in C(a,b) \cap L(a,b)$, then for all $m \in [a,b]$ and $x \in (a,b)$, the following equality holds:*

$$A_{mx}^{0,\lambda}\left\{\int_m^x B_{mt}^{1,\lambda}\left[f(t)\right] dt\right\} = \int_m^x f(t) J_0\left[\lambda(x-t)\right] dt. \tag{6}$$

Theorem 3. [23]. *If $f(x) \in C^1(a,b) \cap L(a,b)$, then for all $m \in [a,b]$ and $x \in (a,b)$, the following equality holds:*

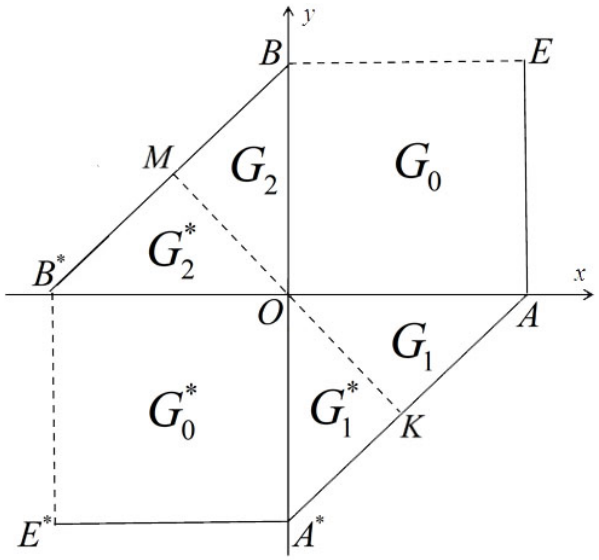
$$A_{mx}^{1,\lambda}\left\{\frac{d}{dx}B_{mx}^{0,\lambda}[f(x)]\right\} = \operatorname{sign}(x-m)C_{mx}^{0,\lambda}[f(x)]\,. \tag{7}$$

Theorem 4. [23]. *If $f(x) \in C^1(a,b) \cap L(a,b)$, then the following equality holds:*

$$\int_k^x f(t)\,J_0\left[\lambda_2\sqrt{(t-k)(t-x)}\right]dt = \int_k^x B_{kx}^{1,\lambda}[f(t)]\,dt, \quad k,\lambda = \text{const}. \tag{8}$$

3. Formulation of the problem

Let G be a finite, simply connected domain in the plane xOy, bounded by the following line segments: $AA^* = \{(x,y) : x - y = 1, 0 \leqslant x \leqslant 1\}$, $A^*E^* = \{(x,y) : y = -1, -1 \leqslant x \leqslant 0\}$, $E^*B^* = \{(x,y) : x = -1, -1 \leqslant y \leqslant 0\}$, $B^*B = \{(x,y) : x - y = -1, 0 \leqslant y \leqslant 1\}$, $BE = \{(x,y) : y = 1, 0 \leqslant x \leqslant 1\}$, $EA = \{(x,y) : x = 1, 0 \leqslant y \leqslant 1\}$.



Let us introduce notations: $D = G \cap (x+y>0)$, $G_0 = D \cap (x>0) \cap (y>0)$, $G_1 = D \cap (y<0)$, $G_2 = D \cap (x<0)$, D^* and the domains G_0^*, G_1^*, G_2^*, symmetric to the domains D and G_0, G_1, G_2 respectively, with respect to the line segment $MK = \{(x,y) : x + y = 0, -1 \leqslant x \leqslant 1\}$;

$$\theta_0(t) = \theta_0(t/2, -t/2), \quad \theta_1(t) = \theta_1((t+1)/2, -(t-1)/2),$$

$$\theta_2(t) = \theta_2(-t/2, t/2), \quad \theta_3(t) = \theta_3((t-1)/2, -(t+1)/2).$$

In the domain G we consider the following equation

$$Lu(x,y) = 0, \tag{9}$$

where

$$L \equiv \begin{cases} \dfrac{\partial^2}{\partial x^2} - H(-xy)\dfrac{\partial^2}{\partial y^2} - H(xy)\dfrac{\partial}{\partial y} - \lambda^2(\operatorname{sign} y) & in \;\; D \setminus (xy=0), \\ \dfrac{\partial^2}{\partial y^2} - H(-xy)\dfrac{\partial^2}{\partial x^2} + H(xy)\dfrac{\partial}{\partial x} + \lambda^2(\operatorname{sign} x) & in \;\; D^* \setminus (xy=0), \end{cases}$$

λ is a given real number, $H(z)$ is Heaviside function.

Equation (9) in the domain G belongs to mixed type, specifically, in the domains G_0, G_0^*, the equation is of parabolic type, and in the domains G_j, G_j^*, where $j = 1, 2$, the equation is of hyperbolic type. The segments $B^*A = \{(x,0) : -1 < x < 1\}$ and $A^*B = \{(0,y) : -1 < y < 1\}$ are the lines of change of type, MK is the line of discontinuity of the last coefficient of the equation.

Definition 1. A regular solution in the domain $D(D^*)$ of equation (9) will be understood as a function

$$u(x,y) \in C(\overline{D}) \cap C^{2,1}_{x,y}(G_0) \cap C^{2,2}_{x,y}(G_1 \cup G_2)$$

$$\left[u(x,y) \in C(\overline{D^*}) \cap C^{1,2}_{x,y}(G_0^*) \cap C^{2,2}_{x,y}(G_1^* \cup G_2^*)\right],$$

which satisfies equation (9) in the domains G_0, G_1 и G_2 (G_0^*, G_1^* и G_2^*) and at the type-changing lines the following continuous gluing conditions

$$u(t,+0) = u(t,-0), \quad u(+0,t) = u(-0,t), \quad 0 \leqslant t \leqslant 1 \quad (-1 \leqslant t \leqslant 0),$$

$$u_y(t,+0) = u_y(t,-0), \quad u_x(+0,t) = u_x(-0,t), \quad 0 < t < 1 \quad (-1 < t < 0)$$

and such that the functions $u_x(t,0)$, $u_y(t,0)$, $u_x(0,t)$ and $u_y(0,t)$ are bounded as $t \to 0$ and $t \to 1$ ($t \to -1$ and $t \to 0$).

In the domain G, for equation (9), we will study the following problem:

Problem H. Find a function $u(x,y)$ with the following properties:

1) $u(x,y) \in C\left(\bar{D}, \bar{D}^*\right) \cap C^1\left(G \backslash MK\right)$ is a regular solution in the domains D and D^* of the equation (9);

2) on the segment $\overline{MK}$ the following matching conditions are satisfied:

$$\lim_{y \to -x+0} u(x,y) = q \lim_{y \to -x-0} u(x,y) + p_1(x), \quad 0 \leqslant x \leqslant 1/2, \tag{10}$$

$$\lim_{x \to -y+0} u(x,y) = q \lim_{x \to -y-0} u(x,y) + p_2(y), \quad 0 \leqslant y \leqslant 1/2; \tag{11}$$

3) $u(x,y)$ satisfies the following boundary conditions:

$$u|_{\overline{AE}} = \varphi(y), \quad 0 \leqslant y \leqslant 1, \tag{12}$$

$$u|_{\overline{A^*E^*}} = \varphi^*(x), \quad -1 \leqslant x \leqslant 0, \tag{13}$$

$$a_1(y) A^{0,\lambda}_{0y}\{u[\theta_2(y)]\} + b_1(y) A^{0,\lambda}_{-1y}\{u[\theta_1(y)]\} + c_1(y) u(0,y) = d_1(y), \quad (0,y) \in \overline{A^*O}, \tag{14}$$

$$a_2(y) A^{0,\lambda}_{0y}\{u[\theta_2(y)]\} + b_2(y) A^{0,\lambda}_{1y}\{u[\theta_3(y)]\} + c_2(y) u(0,y) = d_2(y), \quad (0,y) \in \overline{OB}, \tag{15}$$

where $q \neq 0$ is a given real number, $\varphi(y)$, $\varphi^*(x)$, $p_j(t)$, $a_j(y)$, $b_j(y)$, $c_j(y)$, $d_j(y)$ ($j = 1,2$) are given functions such that $\varphi(y)$, $\varphi^*(x)$ is continuously differentiable, and $p_j(t)$, $a_j(y)$, $b_j(y)$, $c_j(y)$, $d_j(y)$ are twice continuously differentiable functions on their domain of the definitions, and $a^2{}_j([(-1)^j y]) + b^2{}_j([(-1)^j y]) + c^2{}_j([(-1)^j y]) \neq 0$, $y \in [0,1]$, $j = 1,2$; $A^{0,\lambda}_{0y}$ and $A^{0,\lambda}_{1y}$ are the operators defined by (2).

We will assume that the given functions satisfy the following conditions:

$$\left.\begin{gathered} b_j(0) = 0, \quad a_j[(-1)^j] = 0, \quad j = 1,2; \quad a_1(y) \neq b_1(y), \ y \in [-1,0]; \\ d_2(0)[a_2(0) + c_2(0)]^{-1} - q d_1(0)[a_1(0) + c_1(0)]^{-1} = p_1(0) = p_2(0), \\ d_1(-1)[b_1(-1) + c_1(-1)]^{-1} = \varphi^*(0); \end{gathered}\right\} \tag{16}$$

$$\alpha_j(x) \neq 0, \ \beta'_j(x) \leqslant 0, \ \gamma'_j(x) \geqslant 0, \ x \in [0,1], \ j = 1,2, \tag{17}$$

where

$$\alpha_j(x) = a_j\left[(-1)^j x\right] + b_j\left[(-1)^j x\right] + 2c_j\left[(-1)^j x\right],$$

$$\beta_j(x) = a_j\left[(-1)^j x\right]/\alpha_j(x), \quad \gamma_j(x) = b_j[(-1)^j x]/\alpha_j(x).$$

4. Auxiliary problem Γ

In the study of the problem H, the problem Γ is essentially used, which is the determination of the regular solution in the domain D of the equation (9) that satisfies the conditions (12), (15), and

$$a_1(-x)\,A_{0x}^{0,\lambda}\{u[\theta_0(x)]\}+b_1(-x)\,A_{1x}^{0,\lambda}\{u[\theta_1(x)]\}+$$
$$+c_1(-x)\,u(x,0)=d_0(x),\ (x,0)\in OA, \tag{18}$$

where $d_0(x)$ is a given twice differentiable function.

Let $u(x,y)$ be a solution of Problem Γ. Let us introduce the notations

$$u(x,0)=\tau_1(x),\ 0\leqslant x\leqslant 1;\ u_y(x,0)=\nu_1(x),\ 0<x<1;$$
$$u(0,y)=\tau_2(y),\ 0\leqslant y\leqslant 1;\ u_x(0,y)=\nu_2(y),\ 0<y<1.$$

We assume that $\tau_j(z)\in C[0,1]\cap C^2(0,1)$, $\nu_j(z)\in C^1(0,1)$, and $\tau_j'(z)$ and $\nu_j(z)$ are bounded as $z\to 0$ and $z\to 1$, where $j=1,2$. If we assume that $\tau_j(z)$, $\nu_j(z)$ $(j=1,2)$ are temporarily known functions, then the solution to the equation (9) in the domains G_1 and G_2 can be represented in the following forms [4]

$$u(x,y)=\frac{1}{2}[\tau_1(x+y)+\tau_1(x-y)]+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu_1(t)J_0\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]dt+$$
$$+\frac{1}{4}\lambda_2^2 y\int\limits_{x-y}^{x+y}\tau_1(t)\ \bar{J}_1\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]\ dt,\quad (x,y)\in G_1; \tag{19}$$

$$u(x,y)=\frac{1}{2}[\tau_2(y+x)+\tau_2(y-x)]+\frac{1}{2}\int\limits_{y-x}^{y+x}\nu_2(t)J_0\left[\lambda_2\sqrt{(y-t)^2-x^2}\right]\ dt+$$
$$+\frac{1}{4}\lambda_2^2 x\int\limits_{y-x}^{y+x}\tau_2(t)\ \bar{J}_1\left[\lambda_2\sqrt{(y-t)^2-x^2}\right]dt,\quad (x,y)\in G_2. \tag{20}$$

Using (19), after some computations, it is easy to verify that the following equalities hold

$$u[\theta_0(x)]=\frac{1}{2}\tau_1(0)+\frac{1}{2}B_{0x}^{0,\lambda}[\tau_1(x)]-\frac{1}{2}\int\limits_0^x\nu_1(t)J_0\left[\lambda\sqrt{t(t-x)}\right]dt,$$

$$u[\theta_1(x)]=\frac{1}{2}\tau_1(1)+\frac{1}{2}B_{1x}^{0,\lambda}[\tau_1(x)]-\frac{1}{2}\int\limits_x^1\nu_1(t)J_0\left[\lambda\sqrt{(1-t)(x-t)}\right]dt,$$

where $B_{0x}^{0,\lambda}$ and $B_{1x}^{0,\lambda}$ are the operators defined by (3).

Applying to these equalities operators $A_{0x}^{0,\lambda}$ and $A_{1x}^{0,\lambda}$ respectively, and considering (5), (8) and (6), we obtain

$$A_{0x}^{0,\lambda}\{u[\theta_0(x)]\}=\frac{1}{2}\tau_1(0)\,J_0(\lambda x)+\frac{1}{2}\tau_1(x)-\frac{1}{2}\int\limits_0^x\nu_1(t)J_0[\lambda(x-t)]\,dt,$$

$$A_{1x}^{0,\lambda}\left\{u\left[\theta_1\left(x\right)\right]\right\}=\frac{1}{2}\tau_1\left(1\right)J_0\left[\lambda\left(1-x\right)\right]+\frac{1}{2}\tau_1\left(x\right)-\frac{1}{2}\int\limits_x^1\nu_1\left(t\right)J_0\left[\lambda\left(t-x\right)\right]dt.$$

Taking the last equalities into account and considering introduced notations, from the condition(18), we have

$$\alpha_1\left(x\right)\tau_1\left(x\right)=2d_0\left(x\right)-a_1\left(-x\right)\tau_1\left(0\right)J_0\left(\lambda x\right)-$$

$$-b_1\left(-x\right)\tau_1\left(1\right)J_0\left[\lambda\left(1-x\right)\right]+a_1\left(-x\right)\int\limits_0^x\nu_1\left(t\right)J_0\left[\lambda\left(x-t\right)\right]dt+$$

$$+b_1\left(-x\right)\int\limits_x^1\nu_1\left(t\right)J_0\left[\lambda\left(t-x\right)\right]dt,\quad 0\leqslant x\leqslant 1. \tag{21}$$

Equality (21) is a functional relation between the unknown functions $\tau_1\left(x\right)$ and $\nu_1\left(x\right)$ which is brought from the G_1, on the segment OA.

Similarly, using (20) and the condition (15), one can show the validity of the following functional relation between unknown functions $\tau_2\left(x\right)$ and $\nu_2\left(x\right)$ which is brought from the G_2 on the segment OB:

$$\alpha_2\left(x\right)\tau_2\left(x\right)=2d_2\left(x\right)-a_2\left(x\right)\tau_2\left(0\right)J_0\left(\lambda x\right)-$$

$$-b_2\left(x\right)\tau_2\left(0\right)J_0\left[\lambda\left(1-x\right)\right]+a_2\left(x\right)\int\limits_0^x\nu_2\left(t\right)J_0\left[\lambda\left(x-t\right)\right]dt+$$

$$+b_2\left(x\right)\int\limits_x^1\nu_2\left(t\right)J_0\left[\lambda\left(t-x\right)\right]dt,\qquad 0\leqslant x\leqslant 1, \tag{22}$$

where y is re-denoted by x.

Putting the first $x=0$, and then $x=1$ in (21) and (22) and considering conditions $b_1(0)=b_2(0)=a_1(-1)=a_2(1)=0$, we have

$$\tau_1\left(0\right)=\frac{d_0\left(0\right)}{a_1\left(0\right)+c_1\left(0\right)},\quad \tau_1\left(1\right)=\frac{d_0\left(1\right)}{b_1\left(-1\right)+c_1\left(-1\right)}, \tag{23}$$

$$\tau_2\left(0\right)=\frac{d_2\left(0\right)}{a_2\left(0\right)+b_2\left(0\right)},\qquad \tau_2\left(1\right)=\frac{d_2\left(1\right)}{b_2\left(1\right)+c_2\left(1\right)}.$$

Since $u(x,y)\in C(\overline{D})$, from these equalities it follows that the given functions must satisfy the following compatibility conditions

$$d_2(0)[a_1(0)+c_1(0)]=d_0(0)[a_2(0)+c_2(0)],$$

$$d_0(1)=\varphi(0)[b_1(-1)+c_1(-1)].$$

Under these conditions the values of the function $u(x,y)$ at the points (0,1), (0,0) and (1,0) are uniquely defined by the given functions of the problem Γ.

As a result, the problem Γ is equivalently (in the sense of unique solvability) reduced to the problem $\widetilde{\Gamma}$ that defining a solution $u(x,y)\in C(\overline{G}_0)\cap C^1(G_0\cup OA\cup OB)\cap C^2(G_0)$ in the domain G_0 the following equation

$$u_{xx}-u_y-\lambda^2u=0 \tag{24}$$

satisfying the conditions (12), (21) and (22).

Now, we study problem $\widetilde{\Gamma}$. Let $u(x,y)$ be a solution to the problem $\widetilde{\Gamma}$ for $d_2(x) \equiv d_0(x) \equiv \varphi(y) \equiv 0$. Then, in the domain G_0 the following identity holds

$$0 = u\left(u_{xx} - u_y - \lambda^2 u\right) = \left(uu_x\right)_x - \frac{1}{2}\left(u^2\right)_y - \left(u_x\right)^2 - \lambda^2 u.$$

Integrating this identity in the domain G_0 and applying Green's formula, we have

$$2\iint\limits_{G_0}\left[\left(u_x\right)^2 + \lambda^2 u\right]dxdy + \int\limits_0^1 u^2(x,1)\,dx + 2\int\limits_0^1 \tau_2(x)\,\nu_2(x)\,dx = \int\limits_0^1 \tau_1^2(x)\,dx. \quad (25)$$

On the other hand, passing to the limit at $y \to 0$ and considering introduced notations, from (24), we obtain

$$\tau_1''(x) - \lambda^2\tau_1(x) - \nu_1(x) = 0, \quad 0 < x < 1. \quad (26)$$

Multiplying (26) by $\tau_1(x)$ and integrating obtained equality over the segment $[0,1]$, and taking $\tau_1(0) = \tau_1(1) = 0$ into account, we have

$$\int\limits_0^1 \tau_1(x)\,\nu_1(x)\,dx = -\lambda^2\int\limits_0^1 \tau_1^2(x)\,dx - \int\limits_0^1 \left[\tau_1'(x)\right]^2 dx \leqslant 0. \quad (27)$$

Now, using (21) and (22) for $d_2(x) \equiv d_0(x) \equiv \varphi(y) \equiv 0$, we will prove that

$$l_j = \int\limits_0^1 \tau_j(x)\,\nu_j(x)\,dx \geqslant 0, \quad j = 1,2.$$

Let $j = 1$. Then, substituting the representation (21) of $\tau_1(x)$ into l_1, we get

$$l_1 = \int\limits_0^1 \nu_1(x)\,dx\left\{\beta_1(x)\int\limits_0^x \nu_1(t)\,J_0\left[\lambda(x-t)\right]dt + \gamma_1(x)\int\limits_x^1 \nu_1(t)\,J_0\left[\lambda(t-x)\right]dt\right\}.$$

Hence, replacing the function $J_0(z)$ by the following formula (1), we have

$$l_1 = \frac{1}{\pi}\int\limits_{-1}^1 \left(1-\xi^2\right)^{-1/2} d\xi \int\limits_0^1 \nu_1(x)\left[\beta_1(x)\int\limits_0^x \nu_1(t)\cos\left[\lambda(x-t)\right]dt + \right.$$
$$\left. + \gamma_1(x)\int\limits_x^1 \nu_1(t)\cos\left[\lambda(t-x)\right]dt\right]dx.$$

Using the difference formula for cosine functions and the following equality

$$f(x)\int\limits_m^x f(t)\,dt = \frac{1}{2}\frac{d}{dx}\left(\int\limits_m^x f(t)\,dt\right)^2,$$

l_1 can be written in the following form

$$l_1 = \frac{1}{2\pi}\int\limits_{-1}^1 \left(1-\xi^2\right)^{-1/2} d\xi \times$$

$$\times\left\{\int_0^1\beta_1(x)\frac{d}{dx}\left[\left(\int_0^x\nu_1(t)\cos(\lambda t\xi)dt\right)^2+\left(\int_0^x\nu_2(t)\sin(\lambda t\xi)dt\right)^2\right]dx-\right.$$

$$\left.-\int_0^1\gamma_1(x)\frac{d}{dx}\left[\left(\int_x^1\nu_1(t)\cos(\lambda t\xi)\,dt\right)^2+\left(\int_x^1\nu_1(t)\sin(\lambda t\xi)\,dt\right)^2\right]dx\right\}.$$

Hence, applying the rule of integration by parts with respect to x and considering $a_1(-1)=b_1(0)=0$, we have the following expression

$$l_1=\frac{1}{2\pi}\int_{-1}^1\left(1-\xi^2\right)^{-1/2}d\xi\times$$

$$\times\left\{\int_0^1\gamma_1'(x)\left[\left(\int_x^1\nu_1(t)\cos(\lambda t\xi)\,dt\right)^2+\left(\int_x^1\nu_1\sin(\lambda t\xi)\,dt\right)^2\right]dx-\right.$$

$$\left.-\int_0^1\beta_1'(x)\,(x)\left[\left(\int_0^x\nu_1(t)\cos(\lambda t\xi)\,dt\right)^2+\left(\int_0^x\nu_1(t)\sin(\lambda t\xi)\,dt\right)^2\right]dx\right\}$$

from which by virtue of the condition (17) for $j=1$, it follows that $l_1\geqslant 0$.

Similarly, using (22) and the condition (17) for $j=2$, one can show that $l_2\geqslant 0$.

Since $l_1\geqslant 0$, from (27) follows that $\tau_1'(x)\equiv 0$, i. e. $\tau_1(x)=\text{const}$. And since $\tau_1(x)\in C[0,1]$ and $\tau_1(0)=0$, then $\tau_1(x)\equiv 0$, $x\in[0,1]$.

If we consider $\tau_1(x)\equiv 0$, $x\in[0,1]$ and $l_2\geqslant 0$, then from (25) it follows that $u(x,1)\equiv 0$ and $u_x(x,y)=0$, i. e. $u(x,1)=0$ and $u(x,y)=\phi(y)$, where $\phi(y)$ is an arbitrary differentiable function. For the last two equalities, it follows that $u(x,y)\equiv 0$ in $\overline{G_0}$.

Hence, we can conclude that the problem $\widetilde{\Gamma}$ (as a result problem Γ) can not have more than one solution.

Now, we will prove the existence of the solution to the problem $\widetilde{\Gamma}$. To do this, passing to the limit as $y\to 0$, from (24), we obtain (26). If we temporarily assume that $\nu_1(x)$ is a known function then the solution of (26) satisfying the condition (23) is defined by

$$\tau_1(x)=f(x)+\int_0^1K(x,t)\,\nu_1(t)\,dt,\tag{28}$$

where

$$K(x,t)=\begin{cases}-\operatorname{sh}[\lambda_1(1-t)]\operatorname{sh}(\lambda_1 x)\left[\lambda_1\operatorname{sh}\lambda_1\right]^{-1} & \text{for } 0\leqslant x\leqslant t;\\ -\operatorname{sh}(\lambda_1 t)\operatorname{sh}[\lambda_1(1-x)]\left[\lambda_1\operatorname{sh}\lambda_1\right]^{-1} & \text{for } t\leqslant x\leqslant 1,\end{cases}$$

$$f(x)=x\tau_1(1)+(1-x)\,\tau_1(0)-\lambda^2\int_0^1K(x,t)\left[t\tau_1(1)+(1-t)\,\tau_1(0)\right]dt.$$

Excluding the function $\tau_1(x)$ from (21) and (28), we obtain an integral equation of the form with respect to $\nu_1(x)$:

$$\int_0^1K(x,t)\,\nu_1(t)\,dt-\beta_1(x)\int_0^x\nu_1(t)\,J_0[\lambda(x-t)]\,dt-$$

$$-\gamma_1(x)\int_x^1 \nu_1(t)\,J_0[\lambda(t-x)]\,dt = f_1(x), \quad 0<x<1,$$

where

$$f_1(x) = 2\left[d_0(x)/\alpha_1(x)\right] - f(x) - \beta_1(x)\,\tau_1(0)\,J_0(\lambda x) - \gamma_1(x)\,\tau_1(1)\,J_0[\lambda(1-x)].$$

Differentiating this equality with respect to x and considering $\gamma_1(x)-\beta_1(x)\neq 0$, $x\in[0,1]$, we have the following second kind Fredholm integral equation that is equivalent to the problem $\widetilde{\Gamma}$:

$$\nu_1(x) + \int_0^1 K_1(x,t)\,\nu_1(t)\,dt = f_2(x), \quad 0<x<1, \tag{29}$$

where

$$f_2(x) = f_1'(x)\left[\gamma_1(x)-\beta_1(x)\right]^{-1},$$

$$K_1(x,t) = \begin{cases} \left[\gamma_1(x)-\beta_1(x)\right]^{-1}\dfrac{\partial}{\partial x}\left\{K(x,t)-\beta_1(x)\,J_0[\lambda(x-t)]\right\}, & x\geqslant t,\\ \left[\gamma_1(x)-\beta_1(x)\right]^{-1}\dfrac{\partial}{\partial x}\left\{K(x,t)-\gamma_1(x)\,J_0[\lambda(t-x)]\right\}, & x\leqslant t. \end{cases}$$

It is easy to verify that $f_2(x)\in C^1[0,1]$, and the functions $K_1(x,t), K_{1x}(x,t)$ are continuous for $x\neq t$, and for $x=t$ they have a jump of the first kind.

Due to the uniqueness of the solution to the problem $\widetilde{\Gamma}$ and its equivalence to the equation (29), the homogeneous integral equation

$$\nu_1(x) + \int_0^1 K_1(x,t)\nu_1(t)dt = 0, \qquad 0<x<1$$

has only the trivial solution. Then, according to the Fredholm alternative [24], the solution to the non-homogeneous integral equation (29) exists and is unique.

Substituting the function $\nu_1(x)$ found from (29) into (21), we uniquely determine the function $\tau_1(x)$, which belongs to the class $C^2[0,1]$. Thus, the problem $\widetilde{\Gamma}$ is equivalently reduced to the problem of determining the solution of the equation (24) that satisfies the conditions (12), (22), and $u(x,0)=\tau_1(x)$, $0\leqslant x\leqslant 1$.

When solving this problem, we will use the solution formula for the first boundary value problem for equation (24) in the domain G_0:

$$u(x,y) = \int_0^1 e^{-\lambda^2 y}\tau_1(\xi)\,G_1(x,y;\xi,0)\,d\xi + \int_0^y e^{\lambda^2(\eta-y)}\tau_2(\eta)\,G_{1\xi}(x,y;0,\eta)\,dy -$$

$$-\int_0^y e^{\lambda^2(\eta-y)}\varphi(\eta)\,G_{1\xi}(x,y;1,\eta)\,dy, \qquad (x,y)\in G_0, \tag{30}$$

where

$$G_j(x,y;\xi,\eta) = \left[4\pi(y-\eta)\right]^{-1/2}\times$$

$$\times\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi-2n)^2}{\varphi(y-\eta)}\right] + (-1)^j\exp\left[-\frac{(x+\xi-2n)^2}{\varphi(y-\eta)}\right]\right\}, \quad j=1,2.$$

Assuming $x \neq 0$, we differentiate equality (30) with respect to x. Then, taking the limit as $x \to 0$, we obtain

$$\nu_2(y) = f_3(y) - \int_0^y \left[\tau_2(\eta)e^{\lambda^2\eta}\right]' e^{-\lambda^2 y} \left\{[\pi(y-\eta)]^{-1/2} + K_0(y,\eta)\right\} dy, \quad 0 < y < 1, \quad (31)$$

where

$$f_3(y) = e^{-\lambda^2 y}\left\{\int_0^1 \tau_1'(\xi)G_2(0,t;\xi,0)d\xi + \int_0^y \left[\varphi(\eta)e^{-\lambda^2\eta}\right]' G_{2\xi}(0,t;1,\eta)d\eta\right\},$$

$$K_0(y,\eta) = \frac{2}{\sqrt{\pi(y-\eta)}} \sum_{n=1}^{+\infty} \exp\left(\frac{n^2}{\eta - y}\right).$$

Now, in equality (22), we rename x to η and multiply the resulting equality by $\alpha_2^{-1}(\eta)\, e^{\lambda^2\eta}$. Then, differentiating the resulting equality with respect to η, we obtain

$$\begin{aligned}\left[\tau_2(\eta)\, e^{\lambda^2\eta}\right]' &= f_4(\eta) + e^{\lambda^2\eta}\left[\beta_2(\eta) - \gamma_2(\eta)\right]\nu_2(\eta) + \\ &+ \int_0^\eta \nu_2(t)\frac{\partial}{\partial\eta}\left\{e^{\lambda^2\eta}\beta_2(\eta)\, J_0\left[\lambda(\eta - t)\right]\right\} dt + \\ &+ \int_\eta^1 \nu_2(t)\frac{\partial}{\partial\eta}\left\{e^{\lambda^2\eta}\gamma_2(\eta)\, J_0\left[\lambda(y - t)\right]\right\} dt,\end{aligned} \quad (32)$$

where

$$f_4(\eta) = (\partial/\partial\eta)\left\{e^{\lambda^2\eta}\alpha_2^{-1}(\eta)\left[2d_2(\eta) - a_2(\eta)\tau_2(0)J_0(\lambda\eta) - b_2(\eta)\tau_2(1)J_0[\lambda(1-\eta)]\right]\right\}.$$

Substituting (32) into (31), and after some transformations, we obtain a Fredholm integral equation of the second kind, which is equivalent to the problem $\widetilde{\Gamma}$:

$$\nu_2(y) + \int_0^1 \nu_2(t)\, K_2(y,t)\, dt = f_5(y), \quad 0 < y < 1, \quad (33)$$

where

$$f_5(y) = f_3(y) - e^{-\lambda^2 y}\int_0^y f_4(\eta)\left\{[\pi(y-\eta)]^{-1/2} + K_0(y,\eta)\right\} dy,$$

$$K_2(y,t) = e^{\lambda^2(t-y)}\left[\beta_2(t) - \gamma_2(t)\right]\left\{[\pi(y-t)]^{-1/2} + K_0(y,t)\right\} +$$

$$+ \int_t^y \left\{[\pi(y-\eta)]^{-1/2} + K_0(y,\eta)\right\} \cdot \frac{\partial}{\partial\eta}\left\{e^{\lambda^2(\eta-y)}\beta_2(\eta)\, J_0\left[\lambda(\eta - t)\right]\right\} d\eta +$$

$$+ \int_0^t \left\{[\pi(y-\eta)]^{-1/2} + K_0(y,\eta)\right\}\frac{\partial}{\partial\eta}\left\{e^{\lambda^2(\eta-y)}\gamma_2(\eta)\, J_0\left[\lambda(t - \eta)\right]\right\} d\eta, \qquad y \geqslant t,$$

$$K_2(y,t)=\int\limits_0^y\left\{\left[\pi(y-\eta)\right]^{-1/2}+K_0(y,\eta)\right\}\frac{\partial}{\partial\eta}\left\{e^{\lambda^2(\eta-y)}\gamma_2(\eta)J_0\left[\lambda(t-\eta)\right]\right\}d\eta,\ y\leqslant t.$$

The unique and unconditional solvability of the non-homogeneous integral equation (33) (due to its equivalence to the problem $\widetilde{\Gamma}$) follows from the uniqueness of the solution to the problem $\widetilde{\Gamma}$.

After the function $\nu_2(y)$ is found from equation (33), substituting it into equality (22) gives the function $\tau_2(y)$. Therefore, the functions $\tau_j(y)$ and $\nu_j(y)$, $j=1,2$, are uniquely determined through the given problem Γ. Using the found functions $\tau_j(y)$ and $\nu_j(y)$, for $j=1,2$ the solution to the problem Γ in the domains G_0, G_1 and G_2 is determined by the formulas (30), (19), and (20), respectively. This completes the investigation of the problem Γ.

Remark 1. If in the conditions (15) and (18) we have $b_1(-x)\equiv c_1(-x)\equiv 0$, $x\in[0,1]$; $b_2(y)\equiv c_2(y)\equiv 0$, $y\in[0,1]$, then due to the invertibility of the operator $A_{0x}^{0,\lambda}$, the problem Γ reduces to the Tricomi problem, which consists of finding a regular solution in the domain D to equation (9), satisfying the conditions (12) and $u(x,y)|_{\overline{OK}}=X_1(x)$, $x\in[0,1/2]$ and $u(x,y)|_{\overline{OM}}=X_2(y)$, $y\in[0,1/2]$, where $X_1(x)$ and $X_2(y)$ are given continuous functions from the class $C^2[0,1]$.

5. Study of the problem H

Let $u(x,y)$ be a solution to the problem H. We introduce the following notations:

$$\lim_{x\to-y-0}u(x,y)=X(y),\quad 0\leqslant y\leqslant(1/2),\tag{34}$$

$$a_1(-x)A_{0x}^{0,\lambda}\left\{u\left[\theta_0(x)\right]\right\}+b_1(-x)A_{1x}^{0,\lambda}\left\{u\left[\theta_1(x)\right]\right\}+$$
$$c_1(-x)u(x,0)=Y(x),\ (x,0)\in\overline{OA}.\tag{35}$$

We will assume that $X(y)$ and $Y(x)$ are twice continuously differentiable functions (below we will prove that they indeed possess these properties).

From the boundary conditions and the introduced notations, it is clear that if $X(y)$ and $Y(x)$ are given functions, the solution to the problem H in the domain D is determined as the solution to the boundary value problem $\{(9),\ (12),\ (15),\ (35)\}$ by the formulas (30), (19), (20), and in the domain D^* by the the following formulas of the solution of boundary value problem $\{(9),\ (13),\ (14),\ (34)\}$:

$$u(x,y)=\int\limits_0^1 e^{-\lambda^2 y}\tau_1(\xi)G_1(x,y;\xi,0)\,d\xi+\int\limits_0^y e^{\lambda^2(\eta-y)}\tau_2(\eta)G_{1\xi}(x,y;0,\eta)\,dy-$$

$$-\int\limits_0^y e^{\lambda^2(\eta-y)}\varphi(\eta)G_{1\xi}(x,y;1,\eta)\,dy,\qquad (x,y)\in G_0^*,$$

$$u(x,y)=\frac{1}{2}\left[\tau_2^*(y+x)+\tau_2^*(y-x)\right]+\frac{1}{2}\int\limits_{y-x}^{y+x}\nu_2^*(t)J_0\left[\lambda_2\sqrt{(y-t)^2-x^2}\right]\,dt+$$

$$+\frac{1}{4}\lambda_2^2 x\int\limits_{y-x}^{y+x}\tau_2^*(t)\ \bar{J}_1\left[\lambda_2\sqrt{(y-t)^2-x^2}\right]dt,\quad (x,y)\in G_2^*,$$

$$u(x,y)=\frac{1}{2}\left[\tau_1^*(x+y)+\tau_1^*(x-y)\right]+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu_1^*(t)J_0\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]\,dt+$$

$$+\frac{1}{4}\lambda_2^2 y\int\limits_{x-y}^{x+y}\tau_1^*(t)\;\bar{J}_1\left[\lambda_2\sqrt{(x-t)^2-y^2}\right]dt,\quad (x,y)\in G_1^*, \tag{36}$$

where

$$\begin{array}{cccc}\tau_1(t)=u(t,0), & \tau_2(t)=u(0,t), & 0\leqslant t\leqslant 1;\\ \tau_1^*(t)=u(t,0), & \tau_2^*(t)=u(0,t), & -1\leqslant t\leqslant 0;\\ \nu_1(t)=u_y(t,0), & \nu_2(t)=u_x(0,t), & 0<t<1;\\ \nu_1^*(t)=u_y(t,0), & \nu_2^*(t)=u_x(0,t), & -1<t<0.\end{array}$$

Therefore, we focus on finding the unknown functions $X(y)$ and $Y(x)$.

First, we will find the function $Y(x)$. Using the formulas (19) and (36) and the methodology applied in obtaining equality (21), from the relations (14) and (35), we obtain

$$\alpha_1(x)\tau_1(x)=2Y(x)-\tau_1(0)a_1(-x)J_0(\lambda x)-$$

$$\tau_1(1)b_1(-x)J_0[\lambda(1-x)]+a_1(x)\int\limits_0^x\nu_1(x)J_0[\lambda(x-t)]\,dt+$$

$$+b_1(x)\int\limits_x^1\nu_1(t)J_0[\lambda(x-t)]\,dt, \tag{37}$$

$$\alpha_1(x)\tau_2^*(-x)=2d_1(-x)-\tau_2^*(0)a_1(-x)J_0(\lambda x)-$$

$$-\tau_1^*(-1)b_1(-x)J_0[\lambda(1-x)]-a_1(-x)\int\limits_0^x\nu_1^*(-t)J_0[\lambda(x-t)]\,dt-$$

$$-b_1(-x)\int\limits_x^1\nu_2^*(-t)J_0[\lambda(x-t)]\,dt. \tag{38}$$

Multiplying equality (38) by q and subtracting it term by term from (37), we obtain

$$2Y(x)=2qd_1(-x)+\alpha_1(x)[\tau_1(x)-q{\tau_1}^*(-x)]+$$

$$+a_1(-x)J_0(\lambda x)[\tau_1(0)-q{\tau_2}^*(0)]+b_1(-x)J_0[\lambda(1-x)]\times$$

$$\times[\tau_1(1)-q{\tau_2}^*(-1)]-a_1(-x)\int\limits_0^x[\nu(t)+q{\nu_2}^*(-t)]J_0[\lambda(x-t)]dt-$$

$$-b_1(-x)\int\limits_x^1[\nu(t)+q{\nu_2}^*(-t)]J_0[\lambda(x-t)]dt. \tag{39}$$

Therefore, the function $Y(x)$ is uniquely expressed through the unknown function $\delta(x)=\tau_1(x)-q\tau_2*(-x)$. We will find the function $\delta(x)$.

Equation (9) in the domains G_0 and G_0^* has the forms

$$u_{xx}-u_y-\lambda^2u=0,\quad (x,y)\in G_0, \tag{40}$$

$$u_{yy} + u_x + \lambda^2 u = 0, \quad (x, y) \in G_0^*. \tag{41}$$

Passing to the limit as $y \to +0$ and $x \to -0$ in (40) and (41), accordingly, and taking into account the introduced notations, we obtain

$$\tau_1{}''(x) - \lambda^2 \tau_1(x) - \nu_1'(x) = 0, \quad 0 < x < 1, \tag{42}$$

$$\tau_2{}''(-x) + \lambda^2 \tau_2(-x) + \nu_2{}^*(-x) = 0, \quad 0 < x < 1. \tag{43}$$

Multiplying (43) by q and subtracting it term by term from (42), we find

$$\delta''(x) - \lambda^2 \delta(x) - [\nu_1(x) + q\nu_2{}^*(-x)] = 0, \quad 0 < x < 1.$$

As mentioned above, the solution to the problem H in the domains G_1 and G_1^* s determined by formulas (19) and (36), respectively. Substituting (19) and (36) into condition (10) and taking into account the first condition of the compatibility condition (16), after some transformations, we obtain

$$B_{0x}^{0,\lambda}[\tau_1(x) - q\tau_2^*(-x)] - \int_0^x [\nu_1(t) + q\nu_2^*(-t)] J_0\left[\lambda\sqrt{t(t-x)}\right] dt =$$

$$= 2p_1(x/2) - p_1(0), \quad 0 \leqslant x \leqslant 1,$$

where $B_{0x}^{0,\lambda}$ is an operator defined by (3).

Due to equality (8) and the notation $\delta(x) = \tau_1(x) - q\tau_2^*(-x)$ the last equality can be rewritten as

$$B_{0x}^{0,\lambda}\delta(x) - \int_0^x B_{0t}^{1,\lambda}[\nu_1(t) + q\nu_2^*(-t)]\, dt = 2p_1(x/2) - p_1(0), \quad 0 \leqslant x \leqslant 1.$$

Differentiating this equality with respect to x, and then applying the operator $A_{0x}^{1,\lambda}$, defined by formula (2), taking into account the equalities (5) and (7), we obtain

$$\nu_1(z) + q\nu_2^*(-z) = C_{0z}^{0,\lambda}[\delta(z)] - A_{0z}^{1,z}[p_1'(z/2)], \ 0 < z < 1, \tag{44}$$

where $C_{0x}^{0,\lambda}$ is an operator defined by (4). Substituting (44) into (43), we obtain an integral-differential equation with respect to $\delta(x)$:

$$\delta''(x) - \lambda^2 \delta(x) - C_{0x}^{1,\lambda}[\delta(x)] = -A_{0x}^{1,\lambda}[p'(x/2)], \ 0 < x < 1. \tag{45}$$

From (10), (12), and (13), due to the compatibility condition (16), it follows that

$$\delta(0) = p_1(0), \ \delta(1) = \varphi(0) - q\varphi^*(0). \tag{46}$$

We will prove that the solution to equation (45), satisfying condition (46), exists and is unique

First, we will prove the uniqueness of the solution. To do this, it is enough to show that the homogeneous problem

$$\delta''(x) - \lambda^2 \delta(x) - C_{0x}^{0,\lambda}\delta(x) = 0, \quad x \in (0,1); \quad \delta(0) = \delta(1) = 0 \tag{47}$$

has only the trivial solution. To this end, considering the form of the operator $C_{0x}^{0,\lambda}$, we can rewrite the equation in (47) as

$$\delta''(x) - \delta'(x) - \lambda^2 \delta(x) - \frac{1}{2}\lambda^2 \int_0^x \delta(t)\overline{J}_1[\lambda(x-t)]dt = 0, \ x \in (0,1).$$

Multiply this equation by $\delta(x)$ and integrate with respect to x over the interval $[0,1]$. Then, by integrating by parts the first two integrals and taking into account the conditions $\delta(0)=\delta(1)=0$, we obtain

$$\int_0^1 [\delta'(x)]^2 dx + \lambda^2 \int_0^1 \delta^2(x)\,dx + \frac{1}{2}\lambda^2 \int_0^1 \delta(x)\,dx \int_0^x \delta(t)\,\overline{J_1}[\lambda(x-t)\,dt = 0. \tag{48}$$

Let the last integral in (48) be denoted by l and replace the function $\overline{J_1}(z)$ using the formula (1) with $w=1$:

$$l = \frac{2}{\pi}\int_{-1}^1 \left(1-\xi^2\right)^{1/2} \left\{ \int_0^1 \delta(x)\,dx \int_0^x \delta(t)\cos\left[\lambda(x-t)\right] dt \right\} d\xi.$$

From here, applying the formula for the cosine of a difference and taking into account the equality

$$f(x)\int_0^x f(t)\,dt = \frac{1}{2}\frac{d}{dx}\left[\int_0^x f(t)\,dt\right]^2,$$

we compute the integral with respect to x. As a result, we have

$$l = \frac{1}{\pi}\int_{-1}^1 \left(1-\xi^2\right)^{1/2} \left\{ \left[\int_0^1 \delta(t)\cos(\lambda\xi t)\,dt\right]^2 + \left[\int_0^1 \delta(t)\sin(\lambda\xi t)\,dt\right] \right\} d\xi.$$

Therefore, $l \geqslant 0$. Taking this into account, it follows from (48) that $\delta'(x) \equiv 0$, i. e. $\delta(x) = \text{const}$, $x \in (0,1)$. Since $\delta(x) \in C[0,1]$ and $\delta(0) = \delta(1) = 0$, we have $\delta(x) \equiv 0$, $x \in [0,1]$.

Now, we proceed to prove the existence of a solution to the problem $\{(45), (46)\}$. For this purpose, let us rewrite the equations (45) in the form

$$\delta''(x) - \lambda^2 \delta(x) = f(x), \quad 0 < x < 1,$$

where

$$f(x) = -A_{0\,x}^{1,\lambda}[p'(x/2)] + \delta'(x) + (1/2)\lambda^2 \int_0^x \delta(t)\,\bar{J}_1\left[\lambda(x-t)\right]dt. \tag{49}$$

If we temporarily assume that the right hand-side of the last equation is known function, then it is easy to verify that the solution of this equation satisfying the condition (46) has the following form

$$\delta(x) = f_1(x) + \int_0^1 K(x,t) f(t) dt, \tag{50}$$

where

$$f_1(x) = x\delta(1) + (1-x)\delta(0) + \lambda^2 \int_0^1 K(x,t)[t\delta(1) + (1-t)\delta(0)]dt,$$

and $K(x,t)$ is the Green's function defined by

$$K(x,t)=\begin{cases}-\operatorname{sh}[\lambda(1-t)]\operatorname{sh}(\lambda x)(\lambda\operatorname{sh}\lambda)^{-1} & \text{for } x\leqslant t,\\ -\operatorname{sh}[\lambda(1-x)]\operatorname{sh}(\lambda t)(\lambda\operatorname{sh}\lambda)^{-1} & \text{for } x\geqslant t.\end{cases}$$

Substituting the function $f(x)$ from (49) into (50) and applying the integration by parts formula, after some transformations, we obtain the second-kind Fredholm integral equation with respect to $\delta(x)$, which is equivalent to the problem{(45),(46)}:

$$\delta(x)-\int_0^1 K_1(x,t)\,\delta(t)\,dt=f_2(x),\qquad 0<x<1, \tag{51}$$

where

$$f_2(x)=x\delta(1)+(1-x)\,\delta(0)-\int_0^1 K(x,t)\,A_{0t}^{1,\lambda}[p'(t/2)]dt,$$

$$K_1(x,t)=\frac{\partial}{\partial t}K(x,t)-\frac{1}{2}\lambda^2\int_t^1 K(x,z)\,\overline{J}_1[\lambda(z-t)]dz,$$

such that $f_2(x)\in C^1[0,1]\cap C^2(0,1)$, and $K_1(x,t)$ and its derivatives are continuous for $t\neq x$ and they have a jump of the first kind for $t=x$.

The homogeneous problem (47) corresponds to the homogeneous integral equation:

$$\delta(x)-\int_0^1 K_1(x,t)\,\delta(t)\,dt=0,\qquad 0<x<1. \tag{52}$$

Since the problem (47) has only the trivial solution, the equation (52) also has only the solution $\delta(x)\equiv 0$, $x\in[0,1]$. Then, according to the Fredholm alternative [24], the solution to the integral equation (51) exists and is unique. Moreover, due to the properties of $f_2(x)$ and $K_1(x,t)$, its solution belongs to the class $C^1[0,1]\cap C^2(0,1)$.

Therefore, the function $\delta(x)=\tau_1(x)-q\tau_2^*(-x)$ is uniquely determined by the given data of the problem H and belongs to the class $C[0,1]\cap C^2(0,1)$.

Taking this into account, along with the equalities (44) and (39), we can conclude that the unknown function $Y(x)$ is uniquely determined by the given data of the problem H, and moreover, it belongs to the class $Y(x)\in C[0,1]\cap C^2(0,1)$ и $Y(x)\equiv 0$ for $p_1(x)=\varphi(y)\equiv\varphi^*(x)\equiv d_1(x)\equiv 0$.

After the problem Γ for the equation (9) in the domain D with the conditions {(12), (15), (35)} has been solved, by substituting the obtained solution of this problem into equality (11), we find the unknown function $X(y)$:

$$X(y)=\frac{1}{q}\left[\lim_{x\to -y+0}u(x,y)-p_2(y)\right],\qquad 0\leqslant y\leqslant(1/2).$$

Thus, we have proved the following theorem:

Theorem 5. *Let $\varphi(y)\in C^1[0,1]$, $\varphi^*(x)\in C^1[-1,0]$, $p_j(t)\in C^2[0,1/2]$, $a_j(t)$, $b_j(t)$, $c_j(t)$, $d_j(t)\in C^2[-1,0]$ and let the conditions (16), (17) and the following conditions be satisfied:*

$$a^2{}_j\left([(-1)^j y]\right)+b^2{}_j\left([(-1)^j y]\right)+c^2{}_j\left([(-1)^j y]\right)\neq 0,\quad \forall y\in[0,1],\ j=1,2;$$

$$b_j(0)=0,\quad a_j[(-1)^j]=0,\quad j=\overline{1,2};\quad a_1(y)\neq b_1(y),\ y\in[-1,0].$$

Then Problem H has a unique solution.

References

1. **Gel'fand I.M.** Nekotorye voprosy analiza i differentsial,nykh uravneniy [Some questions of analysis and differential equations]. *Uspekhi matematicheskikh nauk* [Achievements of mathematical sciences], 1959, vol. 14, iss. 3, pp. 3–19. (In Russ.).
2. **Ufliand Y.A.** On the propagation of oscillations in composite electrical lines. *Engineering Physics Journal*, 1964, vol. 7, no. 1, pp. 89–92.
3. **Lykov A.V.** Application of irreversible process thermodynamics methods to the study of heat and mass transfer. *Engineering Physics Journal*, 1965, vol. 9, no. 3, pp. 287–304.
4. **Tikhonov A.N., Samarsky A.A.** *Uravneniya matematicheskoy fiziki* [Equations of mathematical physics]. Moscow, Nauka, 1977. (In Russ.).
5. **Shashkov A.G.** *Sistemno-strukturnyy analiz protsessa teploobmena i yego primeneniye* [System-structural analysis of the heat transfer process and its application]. Moscow, Energoatomizdat, 1983. (In Russ.).
6. **Korzyuk V.I., Lemeshevsky S.V., Matus P.P.** Solvability of the conjugation problem for hyperbolic and parabolic equations with integral-differential boundary conditions at the interface. *Proceedings of the Institute of Mathematics*. NAS of Belarus, Minsk, 2000. Vol. 6, pp. 100–108.
7. **Gaiduk S.I.** Primeneniye metoda konturnogo integrala k resheniyu odnoy zadachi na sopryazheniye uravneniy parabolicheskogo i giperbolicheskogo tipov [Applicaton of the contour integral method to solving of the conjugation problem for hyperbolic and parabolic equations]. *Differentisal'nye uravneniya* [Differential equations], 1965, vol. 1, no. 10, pp. 1366–1382. (In Russ.).
8. **Ladyzhenskaya O.A., Stupjalis L.** Ob uravneniyakh smashannogo tipa [On equations of mixed type]. *Vestnik Leningradskogo universiteta. Ser. Matematika, mekhanika, astronomiya* [Bulletin of Leningrad University. Ser. Mathematics, mechanics, astronomy], 1965, vol. 19, no. 4, pp. 38–46. (In Russ.).
9. **Zolina L.A.** On a boundary value problem for a model equation of hyperbolo-parabolic type. *USSR Computational Mathematics and Mathematical Physics*, 1966, vol. 6, no. 6, pp. 63–78.
10. **Bzhikhatlov Kh.G., Nakhushev A.M.** O granichnoy zadache dlya uravneniya smeshannogo tipa [On a boundary value problem for an equation of mixed type]. *Doklady Akademii nauk SSSR* [Reports of USSR Academy of Sciences], 1968, vol. 183, no. 2, pp. 261–264. (In Russ.).
11. **Karimov E.T.** Frankl-type problem for a mixed type equation with the Caputo fractional derivative. *Lobachevskii Journal of Mathematics*, 2020, vol. 41, pp. 1829–1836.
12. **Ashurov R.R., Kadirkulov B.J., Jalilov M.A.** On a nonlocal problem for a mixed-type equation with a fractional order operator. *Lobachevskii Journal of Mathematics*, 2024, vol. 45, pp. 3220–3232.
13. **Urinov A.K., Mamanazarov A.O.** Unique solvability of a nonlocal problem with shift for a parabolic-hyperbolic equation. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2020, vol. 30, no. 2, pp. 270–289.
14. **Mamanazarov A.O.** A non-local problem for a parabolic-hyperbolic equation with singular coefficients. *Uzbek Mathematical Journal*, 2021, vol. 65, no. 1, pp. 118–136.
15. **Zaitseva N.V.** Nonlocal boundary value problem with an integral condition for a mixed type equation with a singular coefficient. *Differential Equations*, 2021, vol. 57, pp. 210–220.

16. **Urinov A.K., Khalilov K.S.** A nonlocal problem for a third order parabolic-hyperbolic equation with a singular coefficient. *Journal of Siberian Federal University. Mathematics & Physics*, 2022, vol. 15, no. 4, pp. 467–481.
17. **Karimov E.T., Rakhmatullaeva N.A.** On a nonlocal problem for mixed parabolic-hyperbolic type equation with nonsmooth line of type changing. *Asian-European Journal of Mathematics*, 2014, vol. 7, no. 2, p. 1450030.
18. **Berdyshev A., Cabada A., Karimov E.** On the existence of eigenvalues of a boundary value problem with transmitting condition of the integral form for a parabolic-hyperbolic equation. *Mathematics*, 2020, vol. 8, no. 6, p. 1030.
19. **Turmetov B.Kh., Kadirkulov B.J.** On a problem for nonlocal mixed-type fractional order equation with degeneration. *Chaos, Solitons and Fractals*, 2021, vol. 146, p. 110835.
20. **Dzhamalov S.Z., Sipatdinova B.K.** A linear inverse problem for a three-dimensional mixed-type equation of the second kind, second order with semi-nonlocal boundary condition in an unbounded parallelepiped. *Chelyabinsk Physical and Mathematical Journal*, 2024, vol. 9, iss. 3, pp. 471–482.
21. **Nakhushev A.M.** *Uravneniya matematicheskoy biologii* [Equations of mathematical biology]. Moscow, Vysshaya shkola, 1995. (In Russ.).
22. **Kuznetsov V.S.** *Spetsial'nye funktsii* [Special functions], Moscow, Vysshaya shkola, 1965. (In Russ.).
23. **Salakhitdinov M.S., Urinov A.K.** *Krayevye zadachi dlya uravneniy smeshannogo tipa so spectral'nym parametrom* [Boundary value problems for equations of mixed type with a spectral parameter]. Tashkent, Fan, 1997. (In Russ.).
24. **Mikhlin S.G.** *Lektsii po lineynym integral'nym uravneniyam* [Lectures on linear integral equations]. Moscow, Fizmatgiz, 1959. (In Russ.).

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О НЕЛОКАЛЬНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ СМЕШАННОГО ПАРАБОЛО-ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ

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Мы сформулировали и исследовали нелокальную задачу с особыми согласующими условиями и условиями сдвига для параболо-гиперболического уравнения. Используя теорию интегральных уравнений, мы установили единственность решения рассматриваемой задачи.

Ключевые слова: *нелокальная задача, параболо-гиперболическое уравнение, метод интегральных уравнений.*

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