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MULTIPLE EULER TYPE INTEGRAL REPRESENTATIONS FOR THE KAMPÉ DE FÉRIET FUNCTIONS

T.G. Ergashev[1,a], **A. Hasanov**[2,3,b], **T.K. Yuldashev**[4,c]

[1]*National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers", Tashkent, Uzbekistan*
[2]*Romanovskiy Institute of Mathematics, National Academy of Sciences, Tashkent, Uzbekistan*
[3]*Ghent University, Ghent, Belgium*
[4]*Tashkent State University of Economics, Tashkent, Uzbekistan*
[a]*ergashev.tukhtasin@gmail.com,* [b]*anvarhasanov@yahoo.com,* [c]*tursun.k.yuldashev@gmail.com*

By the aid of Appell, Humbert and Bessel functions, the integral representations for a Kampé de Fériet function are found. The validity of integral representations for a Kampé de Fériet function of general form are proved. Conditions, under which these representations are expressed in terms of products of two generalized hypergeometric functions are found. Examples, in which the integral representation of the Kampé de Fériet function containing Appell, Humbert or Bessel functions, are identified.

Keywords: *Kampé de Fériet functions, multiple Euler type integral representations, generalized hypergeometric functions of second order, Bessel function, Appell functions, Humbert functions.*

1. Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two and more variables) is motivated essentially by the fact that solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions [1–3]. Such kinds of functions are often referred as special functions of mathematical physics.

It is known that hypergeometric series $F(a,b;c;z)$ is studied by Leonhard Euler. Appell has defined, in 1880, four series: F_1, F_2, F_3, F_4. All of these series are analogous to Gauss' series $F(a,b;c;z)$. P. Humbert has studied confluent hypergeometric series in two variables.

The four Appell series were unified and generalized by Kampé de Fériet [4]. He defined a general hypergeometric series in two variables. The notation introduced by Kampé de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy [5]. Srivastava and Panda [6] gave the definition of a more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation $F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_m); & (\gamma_n);\end{matrix} x,y\right]$ (see, Equation (14)) and announced some groups of conditions on the parameters under which the Kampé

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de Fériet series converges in a non-empty set. Interesting results in this direction have been obtained in the works [7–14].

Many special functions appear as solutions of differential equations or integrals of elementary functions (for instance, see, [15–19]). In the works [20; 21], some Kampé de Fériet functions $F^{1:2;1}_{1:1;0}$, $F^{1:3;1}_{1:2;0}$ and $F^{0:2;2}_{1:0;1}$, $F^{1:1;0}_{1:0;1}$, $F^{1:1;0}_{1:0;1}$, $F^{1:1;1}_{1:0;1}$, $F^{1:1;1}_{1:0;2}$, $F^{1:1;1}_{0:1;2}$ are studied. Thanks to properties of Kampé de Fériet functions, authors manage to obtain a solution to one boundary value problem for the differential equation in explicit form.

Integral representations are very important in the study of applied problems. For evaluations and extensions of results on Euler type integrals, we refer a paper [22]. Also, in this regard, it is noticed that the general sextic equation can be solved in terms of Kampé de Fériet function (see, [23] and [24]). Therefore, well-known reference books [25–27] are highly respected among applied scientists, in which second-order hypergeometric functions in one and two variables are considered. Hasanov and Ruzhansky in [28] constructed Euler-type integral representations for 205 second order hypergeometric series in three variables. However, there are very few works on integral representations of hypergeometric functions when their order exceeds two. We note only work [29], in which 18 integral representations are constructed for some Kampé de Fériet functions of the fourth order.

In this paper we will obtain the multiple Euler type integral representations for the Kampé de Fériet functions of arbitrary order.

2. Preliminaries

We consider the familiar (Euler's) Gamma function $\Gamma(z)$, defined for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by the following formula

$$\Gamma(z) = \begin{cases} \int\limits_0^\infty t^{z-1}e^{-t}dt & (\mathrm{Re}(z) > 0), \\ \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1}(z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \ n \in \mathbb{N}). \end{cases}$$

The general Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda,\, \nu \in \mathbb{C})$ is defined by thr formula

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \prod\limits_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \ \lambda \in \mathbb{C}), \\ \dfrac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \ \nu = k;\ \ n,k \in \mathbb{N}_0;\ \ 0 \leqslant k \leqslant n), \\ 0 & (\lambda = -n;\ \ \nu = k;\ \ n,k \in \mathbb{N}_0;\ \ k > n), \\ \dfrac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \ \lambda \in \mathbb{C} \setminus \mathbb{Z}), \end{cases}$$

where it is understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Gamma quotient exists. Hence, the following standard notations are used:

$$\mathbb{N} := \{1, 2, 3, ...\}\,,\ \ \mathbb{N}_0 := \mathbb{N} \cup \{0\}\ \ \text{and}\ \ \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0,\, -1,\, -2,\, ...\}\,.$$

Moreover, as usual, the symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$, and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive, and negative real numbers, respectively. The Beta function is defined by the following integral

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}\alpha > 0,\ \mathrm{Re}\beta > 0. \tag{1}$$

The celebrated Gauss hypergeometric function

$$F(a,b;c;z) \equiv F\left[\begin{matrix} a,b; \\ c\,; \end{matrix}\ z\right] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \ c \neq 0,-1,-2,... \tag{2}$$

is contained in the generalized hypergeometric function ${}_pF_q$, involving p numerator parameters, $a_1,...,a_p$, and q denominator parameters, and $b_1,...,b_q$, as a special case.

Following the standard notations and conventions, we define it here as follows [30, p. 182]:

$${}_pF_q\left[\begin{matrix} (a_p)\,; \\ (b_q)\,; \end{matrix}\ z\right] \equiv {}_pF_q\left[(a_p)\,;(b_q)\,;z\right] := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k}\frac{z^k}{k!}.$$

Gauss' series (2) in the present notation has the form

$${}_2F_1(a,b;c;z) \equiv F\left[\begin{matrix} a,b; \\ c\,; \end{matrix}\ z\right] := F(a,b;c;z).$$

The familiar Bessel function

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}$$

also belongs to the class of generalized hypergeometric functions ${}_pF_q$:

$${}_0F_1(-;\alpha;-z) \equiv {}_0F_1\left[\begin{matrix} -; \\ \alpha; \end{matrix}\ -z\right] = \Gamma(\alpha)z^{(1-\alpha)/2}J_{\alpha-1}\left(2\sqrt{z}\right).$$

The great success of the theory of hypergeometric series in one variable is that this theory has stimulated the development of a corresponding theory in two and more variables. Four Appell functions are defined as follows [31]:

$$F_1\left(a,b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{3}$$

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!}x^m y^n, \tag{4}$$

$$F_3\left(a,a',b,b';c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{5}$$

$$F_4\left(a,b;c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!}x^m y^n, \tag{6}$$

where, in all definitions (3)–(6), as usual, the denominator parameters c and c' are neither zero nor a negative integer.

Seven confluent forms of the four Appell series were defined by Humbert in [32], and these confluent hypergeometric series in two variables are denoted by

$$\Phi_1(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^my^n,\ \ |x|<1, \tag{7}$$

$$\Phi_2(\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^my^n, \tag{8}$$

$$\Phi_3(\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^my^n, \tag{9}$$

$$\Psi_1(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_nm!n!}x^my^n,\ \ |x|<1, \tag{10}$$

$$\Psi_2(\alpha;\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n\,m!n!}x^my^n, \tag{11}$$

$$\Xi_1(\alpha,\alpha',\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^my^n,\ \ |x|<1, \tag{12}$$

$$\Xi_2(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\,m!n!}x^my^n,\ \ |x|<1, \tag{13}$$

where the denominator parameters γ and γ' are neither zero, nor a negative integer. Hypergeometric functions, which defined in (7)–(13) are called *Humbert functions*.

Just the Gaussian series $F(a,b;c;z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters. The four Appell series were unified and generalized by Kampé de Fériet in [4] and defined a general hypergeometric series in two variables (see, [33, p. 150, eq.(29)]). The notation introduced by Kampé de Fériet for his double hypergeometric series of of superior order was subsequently abbreviated by Burchnall and Chaundy [5, p. 112]. Srivastava and Panda (see, [6] and also [34, Section 3.1]) gave the definition of the more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation

$$F^{p:q,k}_{l:m,n}\left[\begin{matrix}(a_p):\ (b_q)\ ;(c_k)\,;\\ (\alpha_l):(\beta_m);(\gamma_n)\,;\end{matrix}x,y\right]=F^{p:q,k}_{l:m,n}\left[\begin{matrix}a_1,...,a_p:\ b_1,...,b_q\,;\ c_1,...,c_k;\\ \alpha_1,...,\alpha_l:\beta_1,...,\beta_m;\gamma_1,...,\gamma_n;\end{matrix}x,y\right]=$$
$$=\sum_{r,s=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{14}$$

where $p,q,k,l,m,n\in\mathbb{N}\cup\{0\}$, and for convergence of series we put
(i) $p+q<l+m+1$, $p+k<l+n+1$, $|x|<\infty$, $|y|<\infty$ or
(ii) $p+q=l+m+1$, $p+k=l+n+1$, and

$$\begin{cases}|x|^{1/(p-l)}+|y|^{1/(p-l)}<1, & \text{if } p>l,\\ \max\{|x|,\ |y|\}<1, & \text{if } p\leqslant l.\end{cases}$$

Although the double hypergeometric series defined by (14) reduces to Kampé de Fériet series in the *special* case: $q=k$ and $m=n$, yet it is usually referred in the literature as the Kampé de Fériet series.

3. Integral Representations

Theorem 1. *Let* $p,\ q,\ k,\ l,\ m,\ n,\ K,\ Q \in \mathbb{N}\cup\{0\}$ *and* $A_j,\ B_j \in \mathbb{C},\ j\in\mathbb{N}$. *If*

$$\operatorname{Re}(A_j)>0,\ \operatorname{Re}(B_j)>0,\ j\in\mathbb{N}, \tag{15}$$

then for every K *and* Q *the following integral representation formula*

$$F^{\,p\ :q+N-Q;k+N-K}_{l+N:\ \ m\ \ ;\ \ n}\left[\begin{matrix}(a_p) & :(b_q),(A_{N-Q});(c_k),(B_{N-K}); \\ (\alpha_l),(A_N+B_N): & (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int_0^1\cdots\int_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\times$$

$$\times F^{p:\ q\ ;\ k}_{l:m+Q;n+K}\left[\begin{matrix}(a_p): & (b_q)\ ; & (c_k)\ ; \\ (\alpha_l): & (\beta_m),(A_{N-Q+1,N});(\gamma_n),(B_{N-K+1,N});\end{matrix} X,Y\right]dT_N, \tag{16}$$

is valid, where

$$(A_N+B_N):=(A_1+B_1,...,A_N+B_N);$$

$$(A_{N-Q+1,N}):=(A_{N-Q+1},...,A_N),\ \text{if } Q\in\mathbb{N};\quad (A_{N-Q+1,N}):=\emptyset,\ \text{if } Q=0;$$

$$(B_{N-K+1,N}):=(B_{N-K+1},...,B_N),\ \text{if } K\in\mathbb{N};\quad (B_{N-K+1,N}):=\emptyset,\ \text{if } K=0;$$

$$C_N:=\prod_{j=1}^{N}\left[\frac{\Gamma(A_j+B_j)}{\Gamma(A_j)\Gamma(B_j)}\right],\quad X:=x\cdot\prod_{j=1}^{N}t_j,\quad Y:=y\cdot\prod_{j=1}^{N}(1-t_j),\quad dT_N:=dt_1...dt_N.$$

Proof. The equality (16) follows easily from the definition of the Kampé de Fériet series (14), if we use the formula (1) for calculating the Beta function. □

Corollary 1. *Let the conditions* (15) *be satified. If* $p=0$ *and* $l=0$, *then the Kampé de Fériet function defined in* (14) *can be represented as an integral of the product of two generalized hypergeometric functions*

$$F^{0:q+N-Q;k+N-K}_{N:\ \ m\ \ ;\ \ n}\left[\begin{matrix}- & :(b_q),(A_{N-Q});(c_k),(B_{N-K}); \\ (A_N+B_N): & (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int_0^1\cdots\int_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\times$$

$$\times\, {}_qF_{m+Q}\left[\begin{matrix}(b_q)\ ; \\ (\beta_m),(A_{N-Q+1,N});\end{matrix} X\right]{}_kF_{n+K}\left[\begin{matrix}(c_k)\ ; \\ (\gamma_n),(B_{N-K+1,N});\end{matrix} Y\right]dT_N. \tag{17}$$

We consider some interesting examples in the concrete cases of parameters of the functions:

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix}a & :b,A_1;B_1; \\ A_1+B_1: & \beta\ ;-;\end{matrix} x,y\right]=$$

$$=C_1\int_0^1 t^{A_1-1}(1-t)^{B_1-1}F^{1:1;0}_{0:1;0}\left[\begin{matrix}a:b;-; \\ -:\beta;-;\end{matrix} xt,y(1-t)\right]dt, \tag{18}$$

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix} a & : b_1, b_2; B_1; \\ A_1+B_1 : & \beta \quad ; - ; \end{matrix} x, y\right] =$$
$$= C_1 \int\limits_0^1 t^{A_1-1}(1-t)^{B_1-1} F^{1:2;0}_{0:2;0}\left[\begin{matrix} a : b_1, b_2; -; \\ - : \beta, A_1; -; \end{matrix} xt, y(1-t)\right] dt, \quad (19)$$

$$F^{1:4;4}_{4:0;0}\left[\begin{matrix} a & : b, (A_3); b', (B_3); \\ c, (A_3+B_3) : & - \quad ; \quad - \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_1(a, b, b'; c; X, Y)\, dT_3, \quad (20)$$

$$F^{1:4;4}_{3:1;1}\left[\begin{matrix} a & : b, (A_3); b', (B_3); \\ (A_3+B_3) : & c \quad ; \quad c' \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_2(a, b, b'; c, c'; X, Y)\, dT_3, \quad (21)$$

$$F^{1:3;3}_{3:0;0}\left[\begin{matrix} a & : b, A_1, A_2; b', B_1, B_2; \\ (A_3+B_3) : & - \quad ; \quad - \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_2(a, b, b'; A_3, B_3; X, Y)\, dT_3, \quad (22)$$

$$F^{1:4;3}_{3:1;0}\left[\begin{matrix} a & : b, (A_3); b', B_1, B_2; \\ (A_3+B_3) : & c \quad ; \quad - \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_2(a, b, b'; c, B_3; X, Y)\, dT_3, \quad (23)$$

$$F^{1:3;4}_{3:0;1}\left[\begin{matrix} a & : b, A_1, A_2; b', (B_3); \\ (A_3+B_3) : & - \quad ; \quad c \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_2(a, b, b'; A_3, c; X, Y)\, dT_3, \quad (24)$$

$$F^{0:5;5}_{4:0;0}\left[\begin{matrix} - & : a, b, (A_3); a', b', (B_3); \\ c, (A_3+B_3) : & - \quad ; \quad - \quad ; \end{matrix} x, y\right] =$$
$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right] F_3(a, a', b, b'; c; X, Y)\, dT_3, \quad (25)$$

$$F^{2:3;3}_{3:1;1}\left[\begin{matrix} a,\, b & :(A_3)\,;(B_3)\,; \\ (A_3+B_3): & c\ \ ;\ \ c'\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] F_4\left(a,b;c,c';X,Y\right) dT_3, \quad (26)$$

$$F^{2:2;3}_{3:0;1}\left[\begin{matrix} a,\, b & :A_1, A_2;(B_3)\,; \\ (A_3+B_3): & -\ \ ;\ \ c\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] F_4\left(a,b;A_3,c;X,Y\right) dT_3, \quad (27)$$

$$F^{2:3;2}_{3:1;0}\left[\begin{matrix} a,\, b & :(A_3)\,;B_1, B_2; \\ (A_3+B_3): & c\ \ ;\ \ -\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] F_4\left(a,b;c,B_3;X,Y\right) dT_3, \quad (28)$$

$$F^{2:2;2}_{3:0;0}\left[\begin{matrix} a,\, b & :A_1, A_2;B_1, B_2; \\ (A_3+B_3): & -\ \ ;\ \ -\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] F_4\left(a,b;A_3,B_3;X,Y\right) dT_3, \quad (29)$$

$$F^{1:4;3}_{4:0;0}\left[\begin{matrix} a & :b,(A_3)\,;(B_3)\,; \\ c,(A_3+B_3): & -\ \ ;\ \ -\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_1\left(a,b;c;X,Y\right) dT_3, \quad (30)$$

$$F^{0:4;4}_{4:0;0}\left[\begin{matrix} - & :b,(A_3)\,;b',(B_3)\,; \\ c,(A_3+B_3): & -\ \ ;\ \ -\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_2\left(b,b';c;X,Y\right) dT_3, \quad (31)$$

$$F^{0:4;3}_{4:0;0}\left[\begin{matrix} - & :b,(A_3)\,;(B_3)\,; \\ c,(A_3+B_3): & -\ \ ;\ \ -\ \ ; \end{matrix} x, y\right] =$$
$$= C_3 \int_0^1\int_0^1\int_0^1 \prod_{j=1}^{3}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_3\left(b;c;X,Y\right) dT_3, \quad (32)$$

$$F^{1:4;3}_{3:1;1}\left[\begin{matrix} a & :b,(A_3);(B_3); \\ (A_3+B_3): & c \quad ; \quad c' \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;c,c';X,Y)\,dT_3, \quad (33)$$

$$F^{1:3;2}_{3:0;0}\left[\begin{matrix} a & :b,A_1,A_2;B_1,B_2; \\ (A_3+B_3): & - \quad ; \quad - \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;A_3,B_3;X,Y)\,dT_3, \quad (34)$$

$$F^{1:4;2}_{3:1;0}\left[\begin{matrix} a & :b,(A_3);B_1,B_2; \\ (A_3+B_3): & c \quad ; \quad - \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;c,B_3;X,Y)\,dT_3, \quad (35)$$

$$F^{1:3;3}_{3:0;1}\left[\begin{matrix} a & :b,A_1,A_2;(B_3); \\ (A_3+B_3): & - \quad ; \quad c \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;A_3,c;X,Y)\,dT_3, \quad (36)$$

$$F^{1:3;3}_{3:1;1}\left[\begin{matrix} a & :(A_3);(B_3); \\ (A_3+B_3): & c \quad ; \quad c' \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;c,c';X,Y)\,dT_3, \quad (37)$$

$$F^{1:2;2}_{3:0;0}\left[\begin{matrix} a & :A_1,A_2;B_1,B_2; \\ (A_3+B_3): & - \quad ; \quad - \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;A_3,B_3;X,Y)\,dT_3, \quad (38)$$

$$F^{1:3;2}_{3:1;0}\left[\begin{matrix} a & :(A_3);B_1,B_2; \\ (A_3+B_3): & c \quad ; \quad - \quad ; \end{matrix} x,y\right]=$$
$$=C_3\int_0^1\int_0^1\int_0^1\prod_{j=1}^{3}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;c,B_3;X,Y)\,dT_3, \quad (39)$$

$$F^{1:2;3}_{3:0;1}\left[\begin{matrix} a & :A_1, A_2; (B_3)\,; \\ (A_3+B_3) : & - \;\; ; \;\; c \;\; ; \end{matrix} x, y\right] =$$

$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] \Psi_2 (a; A_3, c; X, Y)\, dT_3, \quad (40)$$

$$F^{0:5;4}_{4:0;0}\left[\begin{matrix} - & :a, b, (A_3)\,; a', (B_3)\,; \\ c, (A_3+B_3) : & - \;\; ; \;\; - \;\; ; \end{matrix} x, y\right] =$$

$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] \Xi_1 (a, a', b; c; X, Y)\, dT_3, \quad (41)$$

$$F^{0:5;3}_{4:0;0}\left[\begin{matrix} - & :a, b, (A_3)\,; (B_3)\,; \\ (A_3+B_3) : & - \;\; ; \;\; - \;\; ; \end{matrix} x, y\right] =$$

$$= C_3 \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 \prod_{j=1}^{3} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] \Xi_2 (a, b; c; X, Y)\, dT_3, \quad (42)$$

$$F^{0:N;N}_{N:0;0}\left[\begin{matrix} - & :(A_N)\,; (B_N)\,; \\ (A_N+B_N) : & \alpha \;\; ; \;\; \beta \;\; ; \end{matrix} -x, -y\right] = C_N \left(\sqrt{x}\right)^{1-\alpha} \left(\sqrt{y}\right)^{1-\beta} \times$$

$$\times \underbrace{\int\limits_0^1 \ldots \int\limits_0^1}_{N\text{times}} \prod_{j=1}^{N} \left[t_j^{A_j-\frac{\alpha+1}{2}} (1-t_j)^{B_j-\frac{\beta+1}{2}} \right] J_{\alpha-1}\left(2\sqrt{X}\right)\, J_{\beta-1}\left(2\sqrt{Y}\right) dT_N, \quad (43)$$

$$F^{\;\;1\;\;:N+1;N+1}_{N+1:\;\;0\;\;;\;\;0}\left[\begin{matrix} a & :b, (A_N)\,; b', (B_N)\,; \\ c, (A_N+B_N) : & - \;\; ; \;\; - \;\; ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int\limits_0^1 \ldots \int\limits_0^1}_{N\text{times}} \prod_{j=1}^{N} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] F_1 (a, b, b'; c; X, Y)\, dT_N, \quad (44)$$

$$F^{1:\,N+1;N+1}_{N:\;\;1\;\;;\;\;1}\left[\begin{matrix} a & :b, (A_N)\,; b', (B_N)\,; \\ (A_N+B_N) : & c \;\; ; \;\; c' \;\; ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int\limits_0^1 \ldots \int\limits_0^1}_{N\text{times}} \prod_{j=1}^{N} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] F_2 (a, b, b'; c, c'; X, Y)\, dT_N, \quad (45)$$

$$F^{1:N;N}_{N:0;0}\left[\begin{matrix} a & :b, (A_{N-1})\,; b', (B_{N-1})\,; \\ (A_N+B_N) : & - \;\; ; \;\; - \;\; ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int\limits_0^1 \ldots \int\limits_0^1}_{N\text{times}} \prod_{j=1}^{N} \left[t_j^{A_j-1} (1-t_j)^{B_j-1} \right] F_2 (a, b, b'; A_N, B_N; X, Y)\, dT_N, \quad (46)$$

$$F^{1:N+1;N}_{N:\ 1\ ;0}\left[\begin{matrix} a & :b,(A_N);b',(B_{N-1}); \\ (A_N+B_N): & c\ ;\ -\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_2\left(a,b,b';c,B_N;X,Y\right)dT_N, \quad (47)$$

$$F^{1:N;N+1}_{N:0;\ 1}\left[\begin{matrix} a & :b,(A_{N-1});b',(B_N); \\ (A_N+B_N): & -\ ;\ c\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_2\left(a,b,b';A_N,c;X,Y\right)dT_N, \quad (48)$$

$$F^{\ 0\ :N+2;N+2}_{N+1:\ 0\ ;\ 0}\left[\begin{matrix} - & :a,b,(A_N);a',b',(B_N); \\ c,(A_N+B_N): & -\ ;\ -\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_3\left(a,a',b,b';c;X,Y\right)dT_N, \quad (49)$$

$$F^{2:N;N}_{N:1;1}\left[\begin{matrix} a,\,b & :(A_N);(B_N); \\ (A_N+B_N): & c\ ;\ c'\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_4\left(a,b;c,c';X,Y\right)dT_N, \quad (50)$$

$$F^{2:N-1;N-1}_{N:\ 0\ ;\ 0}\left[\begin{matrix} a,\,b & :(A_{N-1});(B_{N-1}); \\ (A_N+B_N): & -\ ;\ -\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_4\left(a,b;A_N,B_N;X,Y\right)dT_N, \quad (51)$$

$$F^{2:N;N-1}_{N:1;\ 0}\left[\begin{matrix} a,\,b & :(A_N);(B_{N-1}); \\ (A_N+B_N): & c\ ;\ -\ ; \end{matrix} x,y\right]=$$
$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]F_4\left(a,b;c,B_N;X,Y\right)dT_N, \quad (52)$$

$$F_{N:\ 0\ ;1}^{2:N-1;N}\left[\begin{matrix} a,\, b & :(A_{N-1})\,;\, b',(B_N)\,; \\ (A_N+B_N): & -\ ;\ c\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] F_4\left(a,b;A_N,c;X,Y\right) dT_N, \quad (53)$$

$$F_{N+1:\ 0\ ;0}^{\ 1\ :N+1;N}\left[\begin{matrix} a & :b,(A_N)\,;(B_N)\,; \\ c,(A_N+B_N): & -\ ;\ -\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_1\left(a,b;c;X,Y\right) dT_N, \quad (54)$$

$$F_{N+1:\ 0\ ;\ 0}^{\ 0\ :N+1;N+1}\left[\begin{matrix} - & :b,(A_N)\,;\, b',(B_N)\,; \\ c,(A_N+B_N): & -\ ;\ -\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_2\left(b,b';c;X,Y\right) dT_N, \quad (55)$$

$$F_{N+1:\ 0\ ;0}^{\ 0\ :N+1;N}\left[\begin{matrix} - & :b,(A_N)\,;\,(B_N)\,; \\ c,(A_N+B_N): & -\ ;\ -\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Phi_3\left(b;c;X,Y\right) dT_N, \quad (56)$$

$$F_{N:\ 1\ ;1}^{1:N+1;N}\left[\begin{matrix} a & :b,(A_N)\,;(B_N)\,; \\ (A_N+B_N): & c\ ;\ c'\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Psi_1\left(a,b;c,c';X,Y\right) dT_N, \quad (57)$$

$$F_{N:0;\ \ 0}^{1:N;N-1}\left[\begin{matrix} a & :b,(A_{N-1})\,;\,(B_{N-1})\,; \\ (A_N+B_N): & -\ ;\ -\ ; \end{matrix} x, y\right] =$$

$$= C_N \underbrace{\int_0^1 \dots \int_0^1}_{N\text{times}} \prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right] \Psi_1\left(a,b;A_N,B_N;X,Y\right) dT_N, \quad (58)$$

$$F^{1:N+1;N-1}_{N:\ \ 1\ \ ;\ \ 0}\left[\begin{matrix} a & :b,(A_N);(B_{N-1}); \\ (A_N+B_N): & c\ \ ;\ \ -\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;c,B_N;X,Y)\,dT_N, \quad (59)$$

$$F^{1:N;N}_{N:0;1}\left[\begin{matrix} a & :b,(A_{N-1});(B_N); \\ (A_N+B_N): & -\ \ ;\ \ c\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_1(a,b;A_N,c;X,Y)\,dT_N, \quad (60)$$

$$F^{1:N;N}_{N:1;1}\left[\begin{matrix} a & :(A_N);(B_N); \\ (A_N+B_N): & c\ \ ;\ \ c'\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;c,c';X,Y)\,dT_N, \quad (61)$$

$$F^{1:N-1;N-1}_{N:\ \ 0\ \ ;\ \ 0}\left[\begin{matrix} a & :(A_{N-1});(B_{N-1}); \\ (A_N+B_N): & -\ \ ;\ \ -\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;A_N,B_N;X,Y)\,dT_N, \quad (62)$$

$$F^{1:N;N-1}_{N:1;\ \ 0}\left[\begin{matrix} a & :(A_N);(B_{N-1}); \\ (A_N+B_N): & c\ \ ;\ \ -\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;c,B_N;X,Y)\,dT_N, \quad (63)$$

$$F^{1:N-1;N}_{N:\ \ 0\ \ ;1}\left[\begin{matrix} a & :(A_{N-1});(B_N); \\ (A_N+B_N): & -\ \ ;\ \ c\ \ ; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1...\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}(1-t_j)^{B_j-1}\right]\Psi_2(a;A_N,c;X,Y)\,dT_N, \quad (64)$$

$$F^{\,0\;:N+2;N+1}_{N+1:\;\;0\;\;;\;\;0}\left[\begin{matrix} - & :a,b,(A_N)\,;a',(B_N)\,; \\ c,(A_N+B_N): & -\quad;\quad-\quad; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right]\Xi_1\left(a,a',b;c;X,Y\right)dT_N,\quad(65)$$

$$F^{\,0\;:N+2;N}_{N+1:\;\;0\;\;;0}\left[\begin{matrix} - & :a,b,(A_N)\,;(B_N)\,; \\ c,(A_N+B_N): & -\quad;\quad-\quad; \end{matrix} x,y\right]=$$

$$=C_N\underbrace{\int\limits_0^1\ldots\int\limits_0^1}_{N\text{times}}\prod_{j=1}^{N}\left[t_j^{A_j-1}\left(1-t_j\right)^{B_j-1}\right]\Xi_2\left(a,b;c;X,Y\right)dT_N.\quad(66)$$

Integral representations (17)–(66) are easy to prove using the definition (1) of the Beta function.

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КРАТНЫЕ ИНТЕГРАЛЬНЫЕ ПРЕДСТАВЛЕНИЯ ТИПА ЭЙЛЕРА ДЛЯ ФУНКЦИЙ KAMPÉ DE FÉRIET

Т. Г. Эргашев[1,a], **А. Хасанов**[2,3,b], **Т. К. Юлдашев**[4,c]

[1] *Национальный исследовательский университет «Ташкентский институт инженеров ирригации и механизации сельского хозяйства», Ташкент, Узбекистан*
[2] *Институт математики им. В. И. Романовского АН Узбекистана, Ташкент, Узбекистан*
[3] *Университет Гента, Гент, Бельгия*
[4] *Ташкентский государственный экономический университет, Ташкент, Узбекистан*
[a] *ergashev.tukhtasin@gmail.com*, [b] *anvarhasanov@yahoo.com*, [c] *tursun.k.yuldashev@gmail.com*

С помощью функций Аппеля, Гумберта и Бесселя найдены интегральные представления для функции Kampé de Fériet. Доказана справедливость интегральных представлений для функции Kampé de Fériet общего вида. Найдены условия, при которых эти представления выражаются через произведения двух обобщённых гипергеометрических функций. Приведены примеры, в которых интегральное представление функции Kampé de Fériet содержит функции Аппеля, Гумберта или Бесселя.

Ключевые слова: *функции Kampé de Fériet, кратные интегральные представления типа Эйлера, обобщённые гипергеометрические функции второго порядка, функция Бесселя, функция Апелля, функция Гумберта.*

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Сведения об авторах

Эргашев Тухтасин Гуламжанович, доктор физико-математических наук, доцент, профессор кафедры высшей математики, национальный исследовательский университет «Ташкентский институт инженеров ирригации и механизации сельского хозяйства», Ташкент, Узбекистан; e-mail: ergashev.tukhtasin@gmail.com.

Хасанов Анвар, доктор физико-математических наук, профессор, главный научный сотрудник, Институт математики им. В. И. Романовского АН Узбекистана, Ташкент, Узбекистан; e-mail: anvarhasanov@yahoo.com.

Юлдашев Турсун Камалдинович, доктор физико-математических наук, доцент, профессор кафедры общих и точных дисциплин Ташкентского государственного экономического университета, Ташкент, Узбекистан; e-mail: tursun.k.yuldashev@gmail.com.

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