Chelyabinsk Physical and Mathematical Journal. 2023. Vol. 8, iss. 2. P. 228–237.

DOI: 10.47475/2500-0101-2023-18206

A CHARACTERIZATION OF LOCAL DERIVATIONS ON LOW-DIMENSION JORDAN ALGEBRAS

F.N. Arzikulov[1,2,a], **O.O. Nuriddinov**[2,b]

[1] *V.I. Romanovskiy Institute of Mathematics,*
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
[2] *Andizhan State University, Andizhan, Uzbekistan*
[a] *arzikulovfn@rambler.ru*, [b] *o.nuriddinov86@mail.ru*

We consider local derivations on finite-dimensional Jordan algebras. We developed a technique for the description of the vector space of local derivations on an arbitrary low-dimension Jordan algebra. We also give a description of local derivations on some Jordan algebras of dimension four.

Keywords: *Jordan algebra, derivation, local derivation, nilpotent element.*

Introduction

The present paper is devoted to local derivations on Jordan algebras. The history of local derivations begins with the Gleason — Kahane — Żelazko theorem in [1] and [2], which is a fundamental contribution in the theory of Banach algebras. This theorem asserts that every unital linear functional F on a complex unital Banach algebra A, such that $F(a)$ belongs to the spectrum $\sigma(a)$ of a for every $a \in A$, is multiplicative. In modern terminology this is equivalent to the following condition: every unital linear local homomorphism from a unital complex Banach algebra A into $\mathbb{C}$ is multiplicative. We recall that a linear map T from a Banach algebra A into a Banach algebra B is said to be a local homomorphism if for every a in A there exists a homomorphism $\Phi_a : A \to B$, depending on a, such that $T(a) = \Phi_a(a)$.

Later, in [3], R. Kadison introduces the concept of local derivation and proves that each continuous local derivation from a von Neumann algebra into its dual Banach bemodule is a derivation. B. Jonson [4] extends the above result by proving that every local derivation from a C*-algebra into its Banach bimodule is a derivation. In particular, Johnson gives an automatic continuity result by proving that local derivations of a C^*-algebra A into a Banach A-bimodule X are continuous even if not assumed a priori to be so (cf. [4, Theorem 7.5]). Based on these results, many authors have studied local derivations on operator algebras.

By Theorem 5.4 in [5] every local derivation on a JB-algebra is a derivation. So, in [5] the description of local derivations on JB-algebras is given. In [6] the first and the second authors of the present paper have made one of the first contributions to the theory of local mappings in the case of Jordan algebras. They proved that a linear local Jordan multiplier of the Jordan algebra of symmetric matrices over an arbitrary field is a Jordan multiplier operator.

In the present paper, we investigate derivations and local derivations on Jordan algebras. Recall that a linear mapping D on a Jordan algebra $\mathcal{J}$, satisfying, for each pair x, y of elements in $\mathcal{J}$, $D(xy) = D(x)y + xD(y)$, is called a derivation, and a linear mapping $\nabla : \mathcal{J} \to \mathcal{J}$ is called a local derivation if for every $x \in \mathcal{J}$ there exists a derivation $D : \mathcal{J} \to \mathcal{J}$ such that $\nabla(x) = D(x)$.

In the following, we will work over an algebraically closed field $\mathbb{F}$ of characteristic $\neq 2$ and, furthermore, all Jordan algebras are assumed to be of dimension four over $\mathbb{F}$. In [7], the list of Jordan algebras over $\mathbb{F}$ of dimension less than or equal to four is provided.

Description of local derivations of Jordan algebras with nilpotent elements and nilpotent Jordan algebras is an open problem. Therefore, we choose for our investigation appropriate Jordan algebras from this list, and give description of local derivations on these Jordan algebras. We also give a criterion of a linear operator on Jordan algebras of dimension four to be a local derivation. We developed a technique for the description of the vector space of local derivations on an arbitrary low-dimension Jordan algebra.

1. Local derivations on Jordan algebras of dimension four

Let J be a Jordan algebra of dimension four with a basis $\{e_1, e_2, e_3, e_4\}$. Let x be an element in J. Then we can write $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$, for some elements x_1, x_2, x_3, x_4 in $\mathbb{F}$. Throughout of the paper let $\bar{x} = (x_1, x_2, x_3, x_4)^{Tr}$.

Let $T : J \to J$ be a linear operator. Then $T(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} b_{i,j}x_j)e_i$, $x \in J$, for the matrix $B = (b_{i,j})_{i,j=1}^{4}$ of the linear operator T, where $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$. For example, if $J = \mathcal{J}_{62}$ (see Table) and T is a derivation on J, then T has the form $T(x) = a_{1,1}x_1n_1 + 2a_{1,1}x_2n_2 + (a_{3,1}x_1 + 3a_{1,1}x_3)n_3 + a_{1,1}x_4n_4$, $x \in J$, with respect to the basis $\{n_1, n_2, n_3, n_4\}$ (see Table), where $a_{1,1}$, $a_{3,1}$ in $\mathbb{F}$. This common form of a derivation can be directly calculated, and we will omit the calculations of common forms of derivations on the Jordan algebras.

Local derivations of four-dimensional Jordan algebras

$\mathcal{J}$	Multiplication table	Common form of a derivation	Is each local derivation a derivation?
$\mathcal{J}_{49}$	$n_3^2 = n_2n_3 = n_1$, $e_1n_i = \frac{1}{2}n_i$, $i = 2, 3$	$(\beta x_1 + \alpha x_2)n_1 + (\gamma x_3 + (\gamma - \frac{1}{2}\alpha)x_4)n_2$ $+(\alpha - \gamma)x_4n_3$	+
$\mathcal{J}_{62}$	$n_1^2 = n_4^2 = n_2$, $n_1n_2 = n_3$	$a_{1,1}x_1n_1 + 2a_{1,1}x_2n_2 + (a_{3,1}x_1$ $+3a_{1,1}x_3 + a_{3,4}x_4)n_3 + a_{1,1}x_4n_4$	+
$\mathcal{J}_{63}$	$n_1^2 = n_2$, $n_4^2 = -n_2 - n_3$, $n_1n_2 = n_2n_4 = n_3$	$(a_{1,1}x_1 - \frac{1}{3}a_{1,1}x_4)n_1 + (a_{2,1}x_1$ $+2a_{1,1}x_2 + (-\frac{1}{3}a_{1,1} - a_{2,1})x_4)n_2$ $+(a_{3,1}x_1 + 2a_{2,1}x_2 + \frac{8}{3}a_{1,1}x_3$ $+a_{3,4}x_4)n_3 + (-\frac{1}{3}a_{1,1}x_1 + a_{1,1}x_4)n_4$	−
$\mathcal{J}_{64}$	$n_1^2 = n_2$, $n_4^2 = -n_2$, $n_1n_2 = n_2n_4 = n_3$	$(a_{1,1}x_1 + a_{1,4}x_4)n_1 + (a_{2,1}x_1$ $+2a_{1,1}x_2 - a_{2,1})x_4)n_2 + (a_{3,1}x_1$ $+2a_{2,1}x_2 + (3a_{1,1} + a_{1,4})x_3$ $+a_{3,4}x_4)n_3 + (a_{1,4}x_1 + a_{1,1}x_4)n_4$	−
$\mathcal{J}_{65}$	$n_1^2 = n_2$, $n_1n_2 = n_2n_4 = n_3$	$a_{1,1}x_1n_1 + 2a_{1,1}x_2n_2$ $+(a_{3,1}x_1 + a_{3,3}x_3 + a_{3,4}x_4)n_3$ $+((-3a_{1,1} + a_{3,3})x_1 + (-2a_{1,1}$ $+a_{3,3})x_4)n_4$	−

Our principal tool for the description of local derivations on Jordan algebras of dimension four is the common form of derivations, depending on a basis of these Jordan algebras. Our main goal in this paper to justify Table and prove the theorems corresponding to these tables. In Table a necessary and sufficient condition for a linear operator to be a derivation on some Jordan algebras of dimension four is listed. Also, it is indicated that, wether each local derivation of these Jordan algebras is a derivation or not. The third column of the tables indicates whether each local derivation of the corresponding Jordan algebra is a derivation or not, i. e., if yes, then sign "+" is put in the appropriate place of the column, if not, then sign "$-$" is put in this place. All notations of Table are taken from [8].

2. Description of local derivations on some Jordan algebras of dimension four

We prove the appropriate statements for Table, i. e. for four-dimensional Jordan algebras.

Theorem 1. *Each local derivation of the Jordan algebras $\mathcal{J}_{49}$ and $\mathcal{J}_{62}$ is a derivation.*

Proof. Let $\mathcal{J}_{49}$ be the Jordan algebra over the field $\mathbb{F}$ with the basis $\{e_1, n_1, n_2, n_3\}$. Let ∇ be a local derivation on $\mathcal{J}_{49}$. Then

$$\nabla(x) = \left(\sum_{j=1}^{4} b_{1,j}x_j\right) e_1 + \sum_{i=2}^{4}\left(\sum_{j=1}^{4} b_{i,j}x_j\right) n_{i-1}, \quad x \in \mathcal{J}_{49}$$

for the matrix $B = (b_{i,j})_{i,j=1}^4$ of the local derivation ∇, where $x = x_1e_1+x_2n_1+x_3n_2+x_4n_3$ and x_1, x_2, x_3, $x_4 \in \mathbb{F}$.

By the definition for any element $x \in \mathcal{J}_{49}$ there exists a derivation D_x such that $\nabla(x) = D_x(x)$. By the form of a derivation we get

$$D_x(x) = (\beta^x x_1 + \alpha^x x_2)n_1 + \left(\gamma^x x_3 + \left(\gamma^x - \frac{1}{2}\alpha^x\right) x_4\right) n_2 + (\alpha^x - \gamma^x)x_4n_3$$

for some elements α^x, β^x and γ^x in $\mathbb{F}$, depending on x.

Now, by the equality $\nabla(n_1 + n_3) = \nabla(n_1) + \nabla(n_3)$, we have $\alpha^{n_1+n_3} = \alpha^{n_1}$, $\gamma^{n_1+n_3} - \frac{1}{2}\alpha^{n_1+n_3} = \gamma^{n_3} - \frac{1}{2}\alpha^{n_3}$, $\alpha^{n_1+n_3} - \gamma^{n_1+n_3} = \alpha^{n_3} - \gamma^{n_3}$. Hence, $\alpha^{n_1+n_3} = \alpha^{n_3}$ and $\alpha^{n_1} = \alpha^{n_3}$. Thus,

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta^{e_1} & \alpha^{n_1} & 0 & 0 \\ 0 & 0 & \gamma^{n_2} & \gamma^{n_3} - \frac{1}{2}\alpha^{n_1} \\ 0 & 0 & 0 & \alpha^{n_1} - \gamma^{n_3} \end{pmatrix}.$$

The equality $\overline{D(a)} = \overline{\nabla(a)} = B\bar{a}$, where $\bar{a} = (a_1, a_2, a_3, a_4)^{Tr}$, we can rewrite as the following system of linear equations

$$\begin{cases} a_1\beta + a_2\alpha = a_1\beta^{n_1} + a_2\alpha^{n_1}, \\ a_3\gamma + a_4(\gamma - \frac{1}{2}\alpha) = a_3\gamma^{n_2} + a_4(\gamma^{n_3} - \frac{1}{2}\alpha^{n_1}), \\ a_4(\alpha - \gamma) = a_4(\alpha^{n_1} - \gamma^{n_3}), \end{cases}$$

with the parameters a_i, $i = 1, 2, 3, 4$. We rewrite

$$\begin{cases} a_2\alpha + a_1\beta = a_2\alpha^{n_1} + a_1\beta^{e_1}, \\ -\frac{1}{2}a_4\alpha + (a_3 + a_4)\gamma = a_3\gamma^{n_2} + a_4(\gamma^{n_3} - \frac{1}{2}\alpha^{n_1}), \\ a_4\alpha - a_4\gamma = a_4(\alpha^{n_1} - \gamma^{n_3}). \end{cases}$$

Note, if $a_4 = 0$, then the system of linear equations has a solution for any a_1, a_2, a_3.

Now, suppose $a_4 \neq 0$. Then the system of linear equations has the following form

$$\begin{cases} a_1\beta + a_2\alpha = a_1\beta^{n_1} + a_2\alpha^{n_1}, \\ -\alpha + 2(\frac{a_3}{a_4} + 1)\gamma = 2\frac{a_3}{a_4}\gamma^{n_2} + 2\gamma^{n_3} - \alpha^{n_1}, \\ \alpha - \gamma = \alpha^{n_1} - \gamma^{n_3}. \end{cases} .$$

Hence,

$$\begin{cases} a_1\beta + a_2\alpha = a_1\beta^{n_1} + a_2\alpha^{n_1}, \\ (2\frac{a_3}{a_4}+1)\gamma = 2\frac{a_3}{a_4}\gamma^{n_2} + \gamma^{n_3}, \\ \alpha - \gamma = \alpha^{n_1} - \gamma^{n_3}. \end{cases}.$$

If $2\frac{a_3}{a_4}+1 = 0$, then $2\frac{a_3}{a_4}\gamma^{n_2}+\gamma^{n_3}$ mast be equal to 0 since ∇ is a local derivation, i. e., the system of linear equations must have a solution for every a in $\mathcal{J}_{49}$. Hence, $2\frac{a_3}{a_4}\gamma^{n_2}+\gamma^{n_3} = 0$, if $a_4 = -2a_3$, i. e., $\gamma^{n_2} = \gamma^{n_3}$. Thus, by the form of a derivation, we get the statement of the theorem.

The appropriate statement of the theorem for the algebra $\mathcal{J}_{62}$ is similarly proved. The proof is complete. □

Let ∇ be a local derivation on $\mathcal{J}_{63}$. Then $\nabla(x) = \sum_{i=1}^4(\sum_{j=1}^4 b_{i,j}x_j)n_i$, $x \in \mathcal{J}_{63}$ for the matrix $B = (b_{i,j})_{i,j=1}^4$ of the local derivation ∇, where $x = x_1n_1 + x_2n_2 + x_3n_3 + x_4n_4$ and x_1, x_2, x_3, $x_4 \in \mathbb{F}$.

By the definition for any element $x \in \mathcal{J}_{63}$ there exists a derivation D_x such that $\nabla(x) = D_x(x)$. By the form of a derivation we get

$$\overline{D_x(x)} = A_x\bar{x} = \begin{pmatrix} a_{1,1}^x & 0 & 0 & -\frac{1}{3}a_{1,1}^x \\ a_{2,1}^x & 2a_{1,1}^x & 0 & -\frac{1}{3}a_{1,1}^x - a_{2,1}^x \\ a_{3,1}^x & 2a_{2,1}^x & \frac{8}{3}a_{1,1}^x & a_{3,4}^x \\ -\frac{1}{3}a_{1,1}^x & 0 & 0 & a_{1,1}^x \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Similarly to the above algebras we get the following equality

$$B = \begin{pmatrix} a_{1,1}^{n_1} & 0 & 0 & -\frac{1}{3}a_{1,1}^{n_4} \\ a_{2,1}^{n_1} & 2a_{1,1}^{n_2} & 0 & -\frac{1}{3}a_{1,1}^{n_4} - a_{2,1}^{n_4} \\ a_{3,1}^{n_1} & 2a_{2,1}^{n_2} & \frac{8}{3}a_{1,1}^{n_3} & a_{3,4}^{n_4} \\ -\frac{1}{3}a_{1,1}^{n_1} & 0 & 0 & a_{1,1}^{n_4} \end{pmatrix}.$$

Now, by the equality $\nabla(n_1+n_4) = \nabla(n_1)+\nabla(n_4)$ we have $a_{1,1}^{n_1+n_4} - \frac{1}{3}a_{1,1}^{n_1+n_4} = a_{1,1}^{n_1} - \frac{1}{3}a_{1,1}^{n_4}$, $-\frac{1}{3}a_{1,1}^{n_1+n_4} + a_{1,1}^{n_1+n_4} = -\frac{1}{3}a_{1,1}^{n_1} + a_{1,1}^{n_4}$. Hence, $a_{1,1}^{n_1} - \frac{1}{3}a_{1,1}^{n_4} = -\frac{1}{3}a_{1,1}^{n_1} + a_{1,1}^{n_4}$, $a_{1,1}^{n_1+n_4} = a_{1,1}^{n_1} = a_{1,1}^{n_4}$.

On the other hand, the equality $\nabla(n_1+n_4) = \nabla(n_1)+\nabla(n_4)$ gives us the following equalities $a_{2,1}^{n_1+n_4} - \frac{1}{3}a_{1,1}^{n_1+n_4} - a_{2,1}^{n_1+n_4} = a_{2,1}^{n_1} - \frac{1}{3}a_{1,1}^{n_4} - a_{2,1}^{n_4}$, $-\frac{1}{3}a_{1,1}^{n_1+n_4} = a_{2,1}^{n_1} - \frac{1}{3}a_{1,1}^{n_4} - a_{2,1}^{n_4}$. Since $a_{1,1}^{n_1+n_4} = a_{1,1}^{n_4}$, we have $a_{2,1}^{n_1} = a_{2,1}^{n_4}$.

Now $\nabla(n_1+n_2+n_4) = \nabla(n_1)+\nabla(n_2)+\nabla(n_4)$ gives us $a_{2,1}^{n_1+n_2+n_4} + 2a_{1,1}^{n_1+n_2+n_4} - \frac{1}{3}a_{1,1}^{n_1+n_2+n_4} - a_{2,1}^{n_1+n_2+n_4} = a_{2,1}^{n_1} + 2a_{1,1}^{n_2} - \frac{1}{3}a_{1,1}^{n_4} - a_{2,1}^{n_4}$, $\frac{5}{3}a_{1,1}^{n_1+n_2+n_4} = 2a_{1,1}^{n_2} - \frac{1}{3}a_{1,1}^{n_4}$, and $a_{1,1}^{n_1+n_2+n_4} - \frac{1}{3}a_{1,1}^{n_1+n_2+n_4} = a_{1,1}^{n_1} - \frac{1}{3}a_{1,1}^{n_4} - \frac{1}{3}a_{1,1}^{n_1+n_2+n_4} + a_{1,1}^{n_1+n_2+n_4} = -\frac{1}{3}a_{1,1}^{n_1} + a_{1,1}^{n_4}$, $a_{1,1}^{n_1+n_2+n_4} = a_{1,1}^{n_1} = a_{1,1}^{n_4}$. So, $\frac{5}{3}a_{1,1}^{n_1+n_2+n_4} = \frac{5}{3}a_{1,1}^{n_4} = 2a_{1,1}^{n_2} - \frac{1}{3}a_{1,1}^{n_4}$ and $a_{1,1}^{n_2} = a_{1,1}^{n_4}$. Thus, we get

$$B = \begin{pmatrix} a_{11}^{n_1} & 0 & 0 & -\frac{1}{3}a_{11}^{n_1} \\ a_{21}^{n_1} & 2a_{11}^{n_1} & 0 & -\frac{1}{3}a_{11}^{n_1} - a_{21}^{n_1} \\ a_{31}^{n_1} & 2a_{21}^{n_2} & \frac{8}{3}a_{11}^{n_3} & a_{34}^{n_4} \\ -\frac{1}{3}a_{11}^{n_1} & 0 & 0 & a_{11}^{n_1} \end{pmatrix}.$$

We consider the following system of linear equations

$$\begin{cases} a_1a_{1,1} - \frac{1}{3}a_4a_{1,1} = a_1a_{1,1}^{n_1} - \frac{1}{3}a_4a_{1,1}^{n_1}, \\ a_1a_{2,1} + 2a_2a_{1,1} + a_4(-\frac{1}{3}a_{1,1} - a_{2,1}) = a_1a_{2,1}^{n_1} + 2a_2a_{1,1}^{n_1} + a_4(-\frac{1}{3}a_{1,1}^{n_1} - a_{2,1}^{n_1}), \\ a_1a_{3,1} + 2a_2a_{2,1} + \frac{8}{3}a_3a_{1,1} + a_4a_{3,4} = a_1a_{3,1}^{n_1} + 2a_2a_{2,1}^{n_2} + \frac{8}{3}a_3a_{11}^{n_3} + a_4a_{34}^{n_4}, \\ -\frac{1}{3}a_1a_{11} + a_4a_{1,1} = -\frac{1}{3}a_1a_{11}^{n_1} + a_4a_{1,1}^{n_1}. \end{cases}$$

We rewrite this system as follows

$$\begin{cases} (a_1-\frac{1}{3}a_4)a_{1,1}=(a_1-\frac{1}{3}a_4)a_{1,1}^{n_1}, \\ (-\frac{1}{3}a_1+a_4)a_{1,1}=(-\frac{1}{3}a_1+a_4)a_{1,1}^{n_1}, \\ (2a_2-\frac{1}{3}a_4)a_{1,1}+(a_1-a_4)a_{2,1}=(2a_2-\frac{1}{3}a_4)a_{1,1}^{n_1}+(a_1-a_4)a_{2,1}^{n_1}, \\ \frac{8}{3}a_3a_{1,1}+2a_2a_{2,1}+a_1a_{3,1}+a_4a_{3,4}=\frac{8}{3}a_3a_{1,1}^{n_3}+2a_2a_{2,1}^{n_2}+a_1a_{3,1}^{n_1}+a_4a_{3,4}^{n_4}. \end{cases}$$

The subsystem

$$\begin{cases} (a_1-\frac{1}{3}a_4)a_{1,1}=(a_1-\frac{1}{3}a_4)a_{1,1}^{n_1}, \\ (-\frac{1}{3}a_1+a_4)a_{1,1}=(-\frac{1}{3}a_1+a_4)a_{1,1}^{n_1}, \\ (2a_2-\frac{1}{3}a_4)a_{1,1}+(a_1-a_4)a_{2,1}=(2a_2-\frac{1}{3}a_4)a_{1,1}^{n_1}+(a_1-a_4)a_{2,1}^{n_1}, \end{cases}$$

always has a solution. So, if at least one of a_1, a_4 distinct from zero, then our system of linear equations has a solution.

Suppose that $a_1=0$ and $a_4=0$. Then we get

$$\begin{cases} 2a_2a_{1,1}=2a_2a_{1,1}^{n_1}, \\ \frac{8}{3}a_3a_{1,1}+2a_2a_{2,1}=\frac{8}{3}a_3a_{1,1}^{n_3}+2a_2a_{2,1}^{n_2}. \end{cases}$$

If additionally $a_2=0$, then this system has a solution. Else, if $a_2\neq 0$, then we get

$$\begin{cases} a_{1,1}=a_{1,1}^{n_1}, \\ a_{2,1}=\frac{4}{3}\frac{a_3}{a_2}a_{1,1}^{n_3}+a_{2,1}^{n_2}-\frac{4}{3}\frac{a_3}{a_2}a_{1,1}^{n_1}, \end{cases}$$

i. e., in this case our system also has a solution. But, may be $a_{1,1}^{n_1}\neq a_{1,1}^{n_3}$, what does not allow to ∇ be a derivation.

Thus, in all cases the present system of linear equations has a solution. Hence, the linear mapping ∇, defined by the matrix

$$\begin{pmatrix} a_{1,1}^{n_1} & 0 & 0 & -\frac{1}{3}a_{1,1}^{n_1} \\ a_{2,1}^{n_1} & 2a_{1,1}^{n_1} & 0 & -\frac{1}{3}a_{1,1}^{n_1}-a_{2,1}^{n_1} \\ a_{3,1}^{n_1} & 2a_{2,1}^{n_2} & \frac{8}{3}a_{1,1}^{n_3} & a_{3,4}^{n_4} \\ -\frac{1}{3}a_{1,1}^{n_1} & 0 & 0 & a_{1,1}^{n_1} \end{pmatrix}$$

is a local derivation. So, in particular, if $a_{1,1}^{n_1}\neq a_{1,1}^{n_3}$, then ∇ is a local derivation, which is not a derivation. Thus, we have the following theorem.

Theorem 2. *A linear operator $\nabla:\mathcal{J}_{63}\to\mathcal{J}_{63}$ is a local derivation if and only if the matrix of ∇ has the following form*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & -\frac{1}{3}a_{1,1} \\ a_{2,1} & 2a_{1,1} & 0 & -\frac{1}{3}a_{1,1}-a_{2,1} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ -\frac{1}{3}a_{1,1} & 0 & 0 & a_{1,1} \end{pmatrix}.$$

Example 1. By the arguments above, in the case $\mathbb{F}=\mathbb{R}$, the linear operator $\nabla(x)=2x_2n_3$, $x\in\mathcal{J}_{63}$, where $x=x_1n_1+x_2n_2+x_3n_3+x_4n_4$ and x_1, x_2, x_3, $x_4\in\mathbb{F}$, is a local derivation, which is not a derivation.

Let ∇ be a local derivation on $\mathcal{J}_{64}$. Then $\nabla(x)=\sum_{i=1}^4(\sum_{j=1}^4 b_{i,j}x_j)n_i$, $x\in\mathcal{J}_{64}$ for the matrix $B=(b_{i,j})_{i,j=1}^4$ of the local derivation ∇, where $x=x_1n_1+x_2n_2+x_3n_3+x_4n_4$ and x_1, x_2, x_3, $x_4\in\mathbb{F}$.

By the definition for any element $x \in \mathcal{J}_{64}$ there exists a derivation D_x such that $\nabla(x) = D_x(x)$. By the form of a derivation we get

$$\overline{D_x(x)} = A_x\bar{x} = \begin{pmatrix} a^x_{1,1} & 0 & 0 & a^x_{1,4} \\ a^x_{2,1} & 2a^x_{1,1} & 0 & -a^x_{2,1} \\ a^x_{3,1} & 2a^x_{2,1} & 3a^x_{1,1} + a^x_{1,4} & a^x_{3,4} \\ a^x_{1,4} & 0 & 0 & a^x_{1,1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Similarly to the above algebras we get the following equality

$$B = \begin{pmatrix} a^{n_1}_{1,1} & 0 & 0 & a^{n_4}_{1,4} \\ a^{n_1}_{2,1} & 2a^{n_2}_{1,1} & 0 & -a^{n_4}_{2,1} \\ a^{n_1}_{3,1} & 2a^{n_2}_{2,1} & 3a^{n_3}_{1,1} + a^{n_3}_{1,4} & a^{n_4}_{3,4} \\ a^{n_1}_{1,4} & 0 & 0 & a^{n_4}_{1,1} \end{pmatrix}.$$

Now, we take the equality $\nabla(n_1+n_4) = \nabla(n_1)+\nabla(n_4)$. Then we get $a^{n_1+n_4}_{2,1} - a^{n_1+n_4}_{2,1} = a^{n_1}_{2,1} - a^{n_4}_{2,1}$. Hence, $a^{n_1}_{2,1} = a^{n_4}_{2,1}$.

The equality $\overline{D(a)} = \overline{\nabla(a)} = B\bar{a}$ we can represent as the following system of linear equations

$$\begin{cases} a_1a_{1,1} + a_4a_{1,4} = a_1a^{n_1}_{1,1} + a_4a^{n_4}_{1,4}, \\ a_1a_{2,1} + 2a_2a_{1,1} - a_4a_{2,1} = a_1a^{n_1}_{2,1} + 2a_2a^{n_2}_{1,1} - a_4a^{n_1}_{2,1}, \\ a_1a_{3,1} + 2a_2a_{2,1} + a_3(3a_{1,1} + a_{1,4}) + a_4a_{3,4} = \\ = a_1a^{n_1}_{3,1} + 2a_2a^{n_2}_{2,1} + a_3(3a^{n_3}_{1,1} + a^{n_3}_{1,4}) + a_4a^{n_4}_{3,4}, \\ a_1a_{1,4} + a_4a_{1,1} = a_1a^{n_1}_{1,4} + a_4a^{n_4}_{1,1}, \end{cases} \tag{1}$$

with the parameters a_i, $i = 1, 2, 3, 4$. We rewrite this system in the following form

$$\begin{cases} a_1a_{1,1} + a_4a_{1,4} = a_1a^{n_1}_{1,1} + a_4a^{n_4}_{1,4}, \\ a_4a_{1,1} + a_1a_{1,4} = a_1a^{n_1}_{1,4} + a_4a^{n_4}_{1,1}, \\ 2a_2a_{1,1} + (a_1 - a_4)a_{2,1} = 2a_2a^{n_2}_{1,1} + (a_1 - a_4)a^{n_1}_{2,1}, \\ 3a_3a_{1,1} + a_3a_{1,4} + 2a_2a_{2,1} + a_1a_{3,1} + a_4a_{3,4} = \\ = a_1a^{n_1}_{3,1} + 2a_2a^{n_2}_{2,1} + a_3(3a^{n_3}_{1,1} + a^{n_3}_{1,4}) + a_4a^{n_4}_{3,4}. \end{cases}$$

Suppose $a_1 \neq 0$. Then, if $a_1 = a_4$, then the system has the following form

$$\begin{cases} a_1a_{1,1} + a_1a_{1,4} = a_1a^{n_1}_{1,1} + a_1a^{n_4}_{1,4}, \\ 2a_2a_{1,1} = 2a_2a^{n_2}_{1,1}, \\ 3a_3a_{1,1} + a_3a_{1,4} + 2a_2a_{2,1} + a_1a_{3,1} + a_4a_{3,4} = \\ = a_1a^{n_1}_{3,1} + 2a_2a^{n_2}_{2,1} + a_3(3a^{n_3}_{1,1} + a^{n_3}_{1,4}) + a_4a^{n_4}_{3,4}. \end{cases}$$

Since $a_1a_{3,1} \neq 0$, $a_4a_{3,4} \neq 0$ the system has a solution. Else, if $a_1 \neq a_4$, then the subsystem of linear equations

$$\begin{cases} a_1a_{1,1} + a_4a_{1,4} = a_1a^{n_1}_{1,1} + a_4a^{n_4}_{1,4}, \\ a_4a_{1,1} + a_1a_{1,4} = a_1a^{n_1}_{1,4} + a_4a^{n_4}_{1,1} \end{cases}$$

has a solution. Since $(a_1 - a_4)a_{2,1} \neq 0$ we have the subsystem

$$\begin{cases} a_1a_{1,1} + a_4a_{1,4} = a_1a^{n_1}_{1,1} + a_4a^{n_4}_{1,4}, \\ a_4a_{1,1} + a_1a_{1,4} = a_1a^{n_1}_{1,4} + a_4a^{n_4}_{1,1}, \\ 2a_2a_{1,1} + (a_1 - a_4)a_{2,1} = 2a_2a^{n_2}_{1,1} + (a_1 - a_4)a^{n_1}_{2,1} \end{cases}$$

also has a solution for any a_2, $a_1 \neq 0$ and $a_4 \neq a_1$. Hence, by $a_1a_{3,1} \neq 0$ our system of linear equations has a solution.

Suppose $a_1 = 0$. Then the system has the following form

$$\begin{cases} a_4 a_{1,4} = a_4 a_{1,4}^{n_4}, \\ a_4 a_{1,1} = a_4 a_{1,1}^{n_4}, \\ 2a_2 a_{1,1} - a_4 a_{2,1} = 2a_2 a_{1,1}^{n_2} - a_4 a_{2,1}^{n_1}, \\ 3a_3 a_{1,1} + a_3 a_{1,4} + 2a_2 a_{2,1} + a_4 a_{3,4} = 2a_2 a_{2,1}^{n_2} + a_3(3a_{1,1}^{n_3} + a_{1,4}^{n_3}) + a_4 a_{3,4}^{n_4}. \end{cases}$$

In this case, if $a_4 = 0$, then we have

$$\begin{cases} 2a_2 a_{1,1} = 2a_2 a_{1,1}^{n_2}, \\ 3a_3 a_{1,1} + a_3 a_{1,4} + 2a_2 a_{2,1} = 2a_2 a_{2,1}^{n_2} + a_3(3a_{1,1}^{n_3} + a_{1,4}^{n_3}). \end{cases}$$

If we consider the cases $a_2 = 0$ and $a_2 \neq 0$ separatively, then we easily see that this system of linear equations has a solution.

Now, if $a_4 \neq 0$, then we have

$$\begin{cases} a_{1,4} = a_{1,4}^{n_4}, \\ a_{1,1} = a_{1,1}^{n_4}, \\ 2a_2 a_{1,1} - a_4 a_{2,1} = 2a_2 a_{1,1}^{n_2} - a_4 a_{2,1}^{n_1}, \\ 3a_3 a_{1,1} + a_3 a_{1,4} + 2a_2 a_{2,1} + a_4 a_{3,4} = 2a_2 a_{2,1}^{n_2} + a_3(3a_{1,1}^{n_3} + a_{1,4}^{n_3}) + a_4 a_{3,4}^{n_4}, \end{cases}$$

and this system also has a solution for any a_2, a_3 and $a_4 \neq 0$.

Thus, in all cases, the system of linear equations (1) has a solution. Therefore, the map, generated by the matrix

$$\begin{pmatrix} a_{1,1}^{n_1} & 0 & 0 & a_{1,4}^{n_4} \\ a_{2,1}^{n_1} & 2a_{1,1}^{n_2} & 0 & -a_{2,1}^{n_1} \\ a_{3,1}^{n_1} & 2a_{2,1}^{n_2} & 3a_{1,1}^{n_3} + a_{1,4}^{n_3} & a_{3,4}^{n_4} \\ a_{1,4}^{n_1} & 0 & 0 & a_{1,1}^{n_4} \end{pmatrix}$$

is a local derivation by the form of a derivation. At the same time, if, for example, $a_{1,1}^{n_1} \neq a_{1,1}^{n_2}$, then this local derivation (i. e., ∇) is not a derivation again by the form of a derivation. So, we have the following theorem.

Theorem 3. *A linear operator $\nabla : \mathcal{J}_{64} \to \mathcal{J}_{64}$ is a local derivation, if and only if the matrix of ∇ has the following form*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & a_{1,4} \\ a_{2,1} & a_{2,2} & 0 & -a_{2,1} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{1,4} & 0 & 0 & a_{4,4} \end{pmatrix}.$$

Example 2. By the arguments above, in the case $\mathbb{F} = \mathbb{R}$, the linear operator $\nabla(x) = 2x_2 n_2$, $x \in \mathcal{J}_{64}$, where $x = x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4$, is a local derivation, which is not a derivation.

Let ∇ be a local derivation on $\mathcal{J}_{65}$. Then $\nabla(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} b_{i,j} x_j) n_i$, $x \in \mathcal{J}_{65}$ for the matrix $B = (b_{i,j})_{i,j=1}^{4}$ of the local derivation ∇, where $x = x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4$ and x_1, x_2, x_3, $x_4 \in \mathbb{F}$.

By the definition for any element $x \in \mathcal{J}_{65}$ there exists a derivation D_x such that $\nabla(x) = D_x(x)$. By the form of a derivation we get

$$\overline{D_x(x)} = A_x \bar{x} = \begin{pmatrix} a_{1,1}^x & 0 & 0 & 0 \\ 0 & 2a_{1,1}^x & 0 & 0 \\ a_{3,1}^x & 0 & a_{3,3}^x & a_{3,4}^x \\ -3a_{1,1}^x + a_{3,3}^x & 0 & 0 & -2a_{1,1}^x + a_{3,3}^x \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

By the equalities $\nabla(n_i) = D_{n_i}(n_i)$, $\overline{D_{n_i}(n_i)} = A_{n_i}\bar{n}_i$, $i = 1, 2, 3, 4$ we have

$$B = \begin{pmatrix} a_{1,1}^{n_1} & 0 & 0 & 0 \\ 0 & 2a_{1,1}^{n_2} & 0 & 0 \\ a_{3,1}^{n_1} & 0 & a_{3,3}^{n_3} & a_{3,4}^{n_4} \\ -3a_{1,1}^{n_1} + a_{3,3}^{n_1} & 0 & 0 & -2a_{1,1}^{n_4} + a_{3,3}^{n_4} \end{pmatrix}.$$

Now, by the equality $\nabla(n_1 + n_2) = \nabla(n_1) + \nabla(n_2)$ we have $a_{1,1}^{n_1} = a_{1,1}^{n_2}$.

The equality $\overline{D(a)} = \overline{\nabla(a)} = B\bar{a}$ we can rewrite as the following system of linear equations

$$\begin{cases} a_1 a_{1,1} = a_1 a_{1,1}^{n_1}, \\ 2a_2 a_{1,1} = 2a_2 a_{1,1}^{n_1}, \\ a_1 a_{3,1} + a_3 a_{3,3} + a_4 a_{3,4} = a_1 a_{3,1}^{n_1} + a_3 a_{3,3}^{n_3} + a_4 a_{3,4}^{n_4}, \\ a_1(-3a_{1,1} + a_{3,3}) + a_4(-2a_{1,1} + a_{3,3}) = a_1(-3a_{1,1}^{n_1} + a_{3,3}^{n_1}) + a_4(-2a_{1,1}^{n_4} + a_{3,3}^{n_4}), \end{cases} \tag{2}$$

with the parameters a_i, $i = 1, 2, 3, 4$. We rewrite

$$\begin{cases} a_1 a_{1,1} = a_1 a_{1,1}^{n_1}, \\ a_2 a_{1,1} = a_2 a_{1,1}^{n_1}, \\ (-3a_1 - 2a_4)a_{1,1} + (a_1 + a_4)a_{3,3} = a_1(-3a_{1,1}^{n_1} + a_{3,3}^{n_1}) + a_4(-2a_{1,1}^{n_4} + a_{3,3}^{n_4}), \\ a_1 a_{3,1} + a_3 a_{3,3} + a_4 a_{3,4} = a_1 a_{3,1}^{n_1} + a_3 a_{3,3}^{n_3} + a_4 a_{3,4}^{n_4}. \end{cases}$$

Suppose that $a_1 \neq 0$ and $a_1 + a_4 = 0$. Then $a_4 \neq 0$, $-3a_1 - 2a_4 \neq 0$, and

$$\begin{cases} a_{1,1} = a_{1,1}^{n_1}, \\ 2a_{1,1}^{n_1} - 2a_{1,1}^{n_4} = a_{3,3}^{n_1} + a_{3,3}^{n_4}. \end{cases}$$

Clearly, in this case the system (2) has a solution.

If $a_1 \neq 0$ and $a_1 + a_4 \neq 0$, then

$$\begin{cases} a_{1,1} = a_{1,1}^{n_1}, \\ a_{3,3} = \frac{1}{a_1 + a_4}[a_1 a_{3,3}^{n_1} + a_4(-2a_{1,1}^{n_4} + a_{3,3}^{n_4}) + 2a_4 a_{1,1}^{n_1}], \\ a_1 a_{3,1} + a_3 a_{3,3} + a_4 a_{3,4} = a_1 a_{3,1}^{n_1} + a_3 a_{3,3}^{n_3} + a_4 a_{3,4}^{n_4}. \end{cases}$$

Since $a_1 \neq 0$, i. e., $a_1 a_{3,1} \neq 0$ we have, in this case, the system (2) has a solution.

Now, suppose that $a_1 = 0$. Then we get

$$\begin{cases} a_2 a_{1,1} = a_2 a_{1,1}^{n_1}, \\ -2a_4 a_{1,1} + a_4 a_{3,3} = a_4(-2a_{1,1}^{n_4} + a_{3,3}^{n_4}), \\ a_3 a_{3,3} + a_4 a_{3,4} = a_3 a_{3,3}^{n_3} + a_4 a_{3,4}^{n_4}. \end{cases}$$

In this case, if $a_4 = 0$, then the system (2) is equivalent to the following system

$$\begin{cases} a_2 a_{1,1} = a_2 a_{1,1}^{n_1}, \\ a_3 a_{3,3} = a_3 a_{3,3}^{n_3}, \end{cases}$$

and the system (2) has a solution. Else, if $a_4 \neq 0$, then the system (2) is equivalent to the following system

$$\begin{cases} a_2 a_{1,1} = a_2 a_{1,1}^{n_1}, \\ -2a_{1,1} + a_{3,3} = -2a_{1,1}^{n_4} + a_{3,3}^{n_4}, \\ a_3 a_{3,3} + a_4 a_{3,4} = a_3 a_{3,3}^{n_3} + a_4 a_{3,4}^{n_4}. \end{cases}$$

This system has a solution for any a_2, a_3 and $a_4 \neq 0$.

Thus, in all cases, the system of linear equations (2) has a solution if $2a^{n_1}_{1,1} - 2a^{n_4}_{1,1} = a^{n_1}_{3,3} + a^{n_4}_{3,3}$. Hence, by the form of a derivation, under this condition ∇ is a local derivation with the matrix

$$\begin{pmatrix} a^{n_1}_{1,1} & 0 & 0 & 0 \\ 0 & 2a^{n_1}_{1,1} & 0 & 0 \\ a^{n_1}_{3,1} & 0 & a^{n_3}_{3,3} & a^{n_4}_{3,4} \\ -3a^{n_1}_{1,1} + a^{n_1}_{3,3} & 0 & 0 & -2a^{n_4}_{1,1} + a^{n_4}_{3,3} \end{pmatrix}.$$

At the same time, if, additionally, $a^{n_1}_{1,1} \neq a^{n_4}_{1,1}$, then ∇ is a local derivation, which is not a derivation by the form of a derivation. So, we have the following theorem.

Theorem 4. *A linear operator $\nabla : \mathcal{J}_{65} \to \mathcal{J}_{65}$ is a local derivation if and only if the matrix of ∇ has the following form*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ 0 & 2a_{1,1} & 0 & 0 \\ a_{3,1} & 0 & a_{3,3} & a_{3,4} \\ a_{4,1} & 0 & 0 & a_{4,4} \end{pmatrix}.$$

Example 3. By the arguments above, in the case $\mathbb{F} = \mathbb{R}$, the linear operator $\nabla(x) = x_4 n_4$, $x \in \mathcal{J}_{64}$, where $x = x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4$ and x_1, x_2, x_3, $x_4 \in \mathbb{F}$, is a local derivation, which is not a derivation.

Remark 1. We note that local derivations of an arbitrary low-dimension algebra can be similarly described using a common form of the matrix of derivations on this algebra. A technique for constructing a local derivation, which is not a derivation, developed by us, can be applied to an arbitrary low-dimension algebra, derivations of which have a matrix of common form.

Acknowledgments. The authors wish to thank I. Kaygorodov for his suggestions and useful remarks.

References

1. **Gleason A.** A characterization of maximal ideals. *Journal d'Analyse Mathèmatique*, 1967, vol. 19, pp. 171–172.
2. **Kahane J., Żelazko W.** A characterization of maximal ideals in commutative Banach algebras. *Studia Mathematica*, 1968, vol. 29, pp. 339–343.
3. **Kadison R.** Local derivations. *Journal of Algebra*, 1990, vol. 130, pp. 494–509.
4. **Johnson B.** Local derivations on C^*-algebras are derivations. *Transactions of the American Mathematical Society*, 2001, vol. 353, pp. 313–325.
5. **Ayupov Sh., Kudaybergenov K., Peralta A.M.** A survey on local and 2-local derivations on C^*- and von Neumann algebras. *Topics in Functional Analysis and Algebra. Contemporary Mathematics*, 2016, vol. 672, pp. 73–126.
6. **Arzikulov F., Umrzaqov N.** Local mappings generated by a multiplication on rings of matrices. *Mathematical Notes*, 2020, vol. 107, no. 6, pp. 15–25.
7. **Martin M.E.** Four-dimensional Jordan algebras. *International Journal of Mathematics. Game Theory, and Algebra*, 2013, vol. 20, pp. 41—59.
8. **Kashuba I., Martin M.E.** Deformations of Jordan algebras of dimension four. *Journal of Algebra*, 2014, vol. 399, pp. 277–289.

Article received 16.04.2022.
Corrections received 14.06.2023.

Челябинский физико-математический журнал. 2023. Т. 8, вып. 2. С. 228–237.

УДК 518.517 DOI: 10.47475/2500-0101-2023-18206

ХАРАКТЕРИЗАЦИЯ ЛОКАЛЬНЫХ ДИФФЕРЕНЦИРОВАНИЙ ЙОРДАНОВЫХ АЛГЕБР МАЛОЙ РАЗМЕРНОСТИ

Ф. Н. Арзикулов[1,2,a], **О. О. Нуриддинов**[2,b]

[1]*Институт математики имени В. И. Романовского Академии наук Узбекистана, Ташкент, Узбекистан*
[2]*Андижанский государственный университет, Андижан, Узбекистан*
[b]*arzikulovfn@rambler.ru*, [b]*o.nuriddinov86@mail.ru*

Исследуются локальные дифференцирования на конечномерных йордановых алгебрах. Разработана техника для описания пространства локальных дифференцирований на произвольной йордановой алгебре малой размерности. Дано описание локальных дифференцирований на некоторых йордановых алгебрах размерности четыре.

Ключевые слова: *йорданова алгебра, дифференцирование, локальное дифференцирование, нильпотентный элемент.*

Поступила в редакцию 16.04.2022.
После переработки 14.06.2023.

Сведения об авторах

Арзикулов Фарходжон Нематжонович, доктор физико-математических наук, главный научный сотрудник, Институт математики имени В. И. Романовского АН Республики Узбекистан, Ташкент, Узбекистан; Андижанский государственный университет, Андижан, Узбекистан; arzikulovfn@rambler.ru.

Нуриддинов Олимжон Одилжонович, аспирант, Андижанский государственный университет, Андижан, Узбекистан; o.nuriddinov86@mail.ru.