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ASYMPTOTIC EXPANSIONS OF RESONANCES FOR WAVEGUIDES COUPLED THROUGH CONVERGING WINDOWS

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Two-dimensional waveguides coupled through small windows are considered. First terms of the asymptotic expansion of resonances are obtained and studied for the case when the distance between the windows decreases. Method of matching of the asymptotic expansions of solutions of boundary value problems is used.

Keywords: *resonance, asymptotics, coupled waveguides, scattering, low-dimensional system.*

1. Introduction

The problem of resonances for scattering system is intensively studied due to its importance for physical applications (see, e.g., [1–6]. There are several approaches to resonances: Lax — Phillips scattering theory [7], complex scaling [8; 9], perturbation theory [10; 11], functional model [12], asymptotic method [13–15], operator extensions theory model [16–19]. Last decade a new wave of interest to resonance problem for waveguide with perforated boundary and to homogenization is stimulated by nanotechnology progress (see, e. g., [20–24]). For these purposes, it is, particularly, interesting to clarify the resonance behaviour in the case when the distance between the coupling windows vanishes. An additional interest to the question is given by its relation to the choice of regularization when fitting the operator extension theory model [25]. In the present paper, we investigate the problem in the framework of the asymptotic approach.

We consider a system of two plane waveguides Ω^+, Ω^- connected by two apertures (Fig. 1). The wave function satisfies the Helmholtz equation with the Neumann boundary conditions:

$$\Delta u + k^2 u = 0, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0. \tag{1}$$

The lower boundary of the continuous spectrum is zero. The presence of coupling windows leads to appearance of resonances (quasi-eigenvalues) close to the second (third, etc.) threshold value.

Systems of coupled waveguides have been studied for a long time. Estimations and asymptotics for bound states close to the lower bound of the continuous spectrum were obtained in [26; 27]. In the present paper, we use the technique similar to that in [28–31]. The method is based on matching of the asymptotic expansions of the solutions of boundary value problems [32]. Particularly, asymptotic estimates of the resonance (quasi eigenvalue) close to the second threshold π^2/d_+^2 (d_+ is the width of the widest waveguide)

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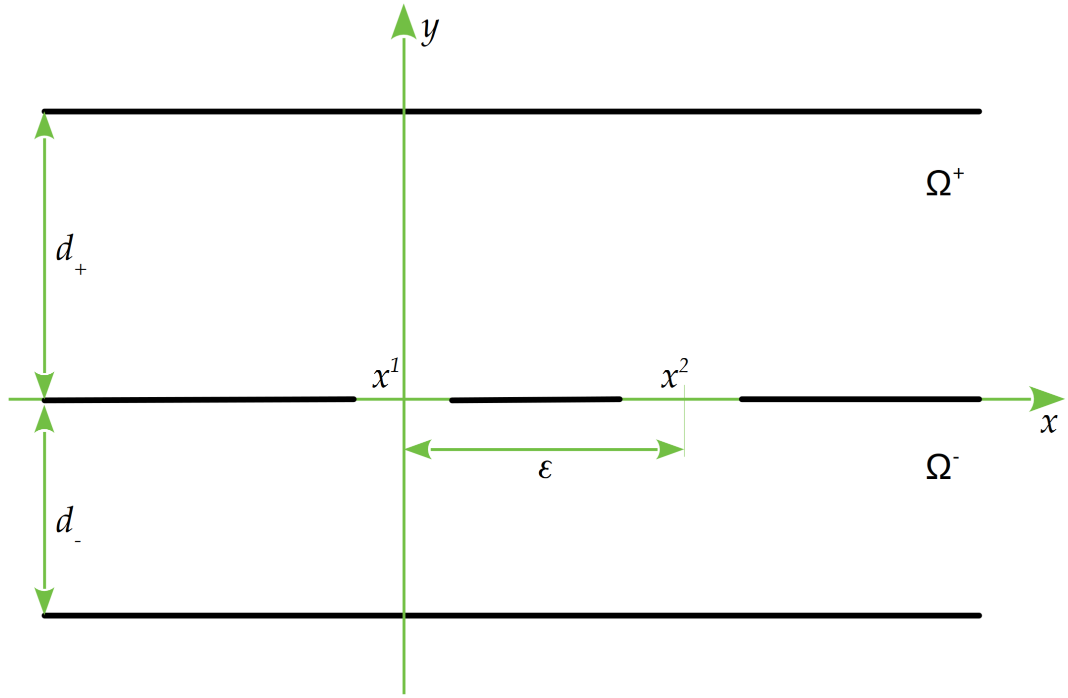


Fig. 1. Geometry of the system

of the continuous spectrum of the Neumann Laplacian for two-dimensional strips coupled through a small window were obtained in [29].

In the present article, we obtain the first terms of the asymptotic expansion of the resonance close to the second threshold assuming that the distance between the openings decreases along with their diameters.

2. The problem description

Let d_+ and d_- be the waveguide widths, $d_+ > d_-$. Let a be small parameter and the radius of the apertures are $a_1 = a\omega_1$, $a_2 = a\omega_2$. Choose the coordinate system as shown in Fig. 1. We assume that the distance ε between the centers $(x^1, 0)$ and $(x^2, 0)$ of the apertures decreases when $a \to 0$ and the character of ε decreasing is related to the decreasing of a:

$$\varepsilon = |x^1 - x^2| = ma^\delta, \quad 0 < \delta < 1. \tag{2}$$

One can note that $\delta = 1$ corresponds to simple similarity in position and size of the apertures.

For resonance value k_a^2 close to the second threshold π^2/d_+^2, we construct the following asymptotic expansion in the form of series in powers of $\ln^{-1} a$

$$\gamma_a = \sqrt{\frac{\pi^2}{d_+^2} - k_a^2} = \tau_1 \ln^{-1} a + \tau_2 \ln^{-2} a + \ldots \tag{3}$$

Our goal is to find the first coefficients τ_1 and τ_2 of this asymptotic expansion (3).

3. Construction of asymptotic expansion

To find a formal asymptotic solution to the problem (1), we need the expression for the Green function in a single waveguide with the Neumann boundary conditions:

$$G^\pm(X_1, X_2, k) = \sum_{n=0}^{\infty} \frac{1}{d_\pm \gamma_n^\pm (\delta_{n0} + 1)} \cos\frac{\pi n y_1}{d_\pm} \cos\frac{\pi n y_2}{d_\pm} \exp\Big(- \gamma_n^\pm d_\pm |x_1 - x_2|\Big), \tag{4}$$

where $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ are the Cartesian coordinates, δ_{n0} is Kronecker symbol and

$$\gamma_n^{\pm} = \sqrt{\frac{\pi^2 n^2}{d_{\pm}^2} - k^2}.$$

The asymptotic expansions of the Green function (4) in the neighborhood of singularity $(0,0)$ $(x \to x^1 = 0)$ have the form:

$$\begin{aligned} G^{+}\Big((x,0);(x^1,0);k_a\Big) &\sim \frac{1}{d_{+}\sqrt{\pi^2/d_{+}^2 - k_a^2}} - \frac{1}{\pi}\ln|x| + g^{+}(x), \\ G^{-}\Big((x,0);(x^1,0);k_a\Big) &\sim -\frac{1}{\pi}\ln|x| + g^{-}(x), \end{aligned} \tag{5}$$

where functions $g^{\pm}(x_1)$ have no singularities in the waveguide.

We also need another asymptotic behavior of the Green function near the next opening for distance ε (2) between the windows tending to zero:

$$\begin{aligned} G^{+}\Big((x,0);(x^2,0);k_a\Big) &\sim \frac{1}{\gamma_a} - \frac{\delta}{\pi}\ln a + h^{+}(x - x^2), \\ G^{-}\Big((x,0);(x^2,0);k_a\Big) &\sim -\frac{\delta}{\pi}\ln a + h^{-}(x - x^2), \end{aligned}$$

where functions $h^{\pm}(x)$ have no singularities in the waveguide.

The main result of the present paper is as follows.

Theorem 1. *There are two resonances k_a^2 close to the second threshold π^2/d_+^2 which have the following asymptotic expansions in the form of series in powers of $\ln^{-1} a$:*

$$\gamma_a = \sqrt{\frac{\pi^2}{d_+^2} - k_a^2} = \tau_1 \ln^{-1} a + \tau_2 \ln^{-2} a + \dots, \tag{6}$$

where

$$\tau_1 = \frac{\pi}{(1+\delta)d_+},$$

$$\begin{aligned} \tau_2^{+} &= \frac{\pi^2}{d_+(1+\delta)^2}\left(g^{+}(0) + g^{-}(0) + \ln\frac{2}{\omega} - \frac{1}{\pi}\ln m\right), \\ \tau_2^{-} &= \frac{\pi^2}{d_+(1-\delta)^2}\left(\ln\frac{2}{\omega} + \frac{1}{\pi}\ln m\right), \end{aligned}$$

$\omega_1 = \omega_2 = \omega$.

The goal of the paper is proving of the theorem. We will use matching of asymptotic expansions of the corresponding solutions (quasi-eigenfunctions). To construct the asymptotic expansion for quasi-eigenfunction $\psi_a(X)$, we perform a partition of the domain. For this purpose, we introduce four disks. Their centers are the windows centers and their radii are $r(a)$ and $2r(a)$ where $r(a)$ is chosen in such a way that

$$a\omega_1 < r(a) < 2r(a) < \varepsilon/2 = \frac{m}{2}a^{\delta}.$$

Then, we construct the asymptotic expansion for quasi-eigenfunction $\psi_a(X)$ in the form

$$\psi_a(X) = \begin{cases} \pm\gamma_a\Big(\alpha_1 G^{\pm}(X,(x^1,0),k_a) + \alpha_2 G^{\pm}(X,(x^2,0),k_a)\Big), & X \in \Omega^{\pm} \setminus S^{1,2}_{r(a)}; \\ v_0^{1,2}(x/a) + v_1^{1,2}(x/a)\ln^{-1} a + \dots, & X \in S^{1,2}_{2r(a)}. \end{cases} \tag{7}$$

In accordance with the matching method for the leading terms, we need to find such "bonding" functions, satisfying the Neumann boundary conditions, that the terms of the corresponding orders in the asymptotic expansions of the solution coincide in the domains

$$\left(\Omega^+ \setminus S^{1,2}_{r(a)}\right) \cap S^{1,2}_{2r(a)}, \quad \left(\Omega^- \setminus S^{1,2}_{r(a)}\right) \cap S^{1,2}_{2r(a)}.$$

Functions $v_0^{1,2}$ and $v_1^{1,2}$ should satisfy the Laplace equation with the Neumann boundary conditions. Further, to determine the form of these functions, we compare the coefficients for the corresponding powers of a in the expansion (7).

The expansion (7) in the neighborhood of the first window ($x^1 = 0$) takes the form

$$\psi_a(X) = \begin{cases} \gamma_a \Big[\alpha_1 \Big(\frac{1}{d_+ \gamma_a} - \frac{1}{\pi} \ln a - \frac{1}{\pi} \ln |x/a| + g^+(x) \Big) + \\ +\alpha_2 \Big(\frac{1}{d_+ \gamma_a} - \frac{\delta}{\pi} \ln a + h^+(x) \Big) \Big], & X \in \Omega^+ \setminus S_{r(a)}; \\ v_0^1(x/a) + v_1^1(x/a) \ln^{-1} a + \ldots, & X \in S_{2r(a)}; \\ -\gamma_a \Big[\alpha_1 \Big(-\frac{1}{\pi} \ln a - \frac{1}{\pi} \ln |x/a| + g^-(x) \Big) + \\ +\alpha_2 \Big(-\frac{\delta}{\pi} \ln a + h^-(x) \Big) \Big], & X \in \Omega^- \setminus S_{r(a)}. \end{cases} \tag{8}$$

Using (6), we equate the coefficients with a^0 in (8):

$$\left(\frac{1}{d_+} - \frac{\tau_1}{\pi} \right) \alpha_1 + \left(\frac{1}{d_+} - \frac{\delta \tau_1}{\pi} \right) \alpha_2 = \frac{\tau_1}{\pi} \alpha_1 + \frac{\delta \tau_1}{\pi} \alpha_2,$$

and obtain two equations corresponding to $x \to x^1$ and $x \to x^2$:

$$\begin{cases} \left(\frac{1}{d_+} - \frac{2\tau_1}{\pi} \right) \alpha_1 + \left(\frac{1}{d_+} - \frac{2\delta\tau_1}{\pi} \right) \alpha_2 = 0, \\ \left(\frac{1}{d_+} - \frac{2\delta\tau_1}{\pi} \right) \alpha_1 + \left(\frac{1}{d_+} - \frac{2\tau_1}{\pi} \right) \alpha_2 = 0. \end{cases} \tag{9}$$

System (9) should have nonzero solution, then

$$\tau_1 = \frac{\pi}{(1+\delta) d_+}. \tag{10}$$

We choose the constant functions as $v_0^{1,2}$:

$$v_0^1(x/a) \equiv \frac{\tau_1}{\pi} (\alpha_1 + \delta \alpha_2); \quad v_0^2(x/a) \equiv \frac{\tau_1}{\pi} (\delta \alpha_1 + \alpha_2).$$

For finding τ_2, we equate the coefficients with $\ln^{-1} a$ in (8):

$$\begin{cases} \left(-\frac{\tau_1}{\pi} \ln |\frac{x}{a}| + \tau_1 g^+(0) - \frac{\tau_2}{\pi} \right) \alpha_1 + \left(\tau_1 h^+(0) - \frac{\delta \tau_2}{\pi} \right) \alpha_2 = \\ = -\frac{\tau_1}{\pi} \left(\ln |\frac{x}{a}| + \ln \left(\frac{2}{\omega_1} \right) \right) \alpha_1 + C_1, \\ -\left(-\frac{\tau_1}{\pi} \ln |\frac{x}{a}| + \tau_1 g^-(0) - \frac{\tau_2}{\pi} \right) \alpha_1 - \left(\tau_1 h^-(0) + \frac{\delta \tau_2}{\pi} \right) \alpha_2 = \\ = \frac{\tau_1}{\pi} \left(\ln |\frac{x}{a}| + \ln \left(\frac{2}{\omega_1} \right) \right) \alpha_1 + C_1, \end{cases}$$

or

$$\begin{cases} \left(\tau_1 g^+(0) + \frac{\tau_1}{\pi}\ln\left(\frac{2}{\omega_1}\right) - \frac{\tau_2}{\pi}\right)\alpha_1 + \left(\tau_1 h^+(0) - \frac{\delta\tau_2}{\pi}\right)\alpha_2 = C_1, \\ -\left(\tau_1 g^-(0) + \frac{\tau_1}{\pi}\ln\left(\frac{2}{\omega_1}\right) - \frac{\tau_2}{\pi}\right)\alpha_1 - \left(\tau_1 h^-(0) + \frac{\delta\tau_2}{\pi}\right)\alpha_2 = C_1. \end{cases}$$

It is known that there exist suitable functions $v_1^{1,2}$:

$$v_1^{1,2} = \begin{cases} -\frac{\tau_1\alpha_{1,2}}{\pi} X_0^{1,2}(x/a) + C_{1,2}, & x > 0; \\ \frac{\tau_1\alpha_{1,2}}{\pi} X_0^{1,2}(x/a) + C_{1,2}, & x < 0, \end{cases}$$

where

$$X_0^{1,2}(\zeta) = X_0\left(\frac{\zeta}{\omega_{1,2}} - x^{1,2}\right), \quad X_0(\zeta) = \ln\left(\zeta + \sqrt{\zeta^2 - 1}\right),$$

and

$$X_0^{1,2}(\zeta) = \ln|\zeta| + \ln 2 - \ln|\omega_{1,2}| + o(1), \quad \zeta \to +\infty.$$

Then asymptotics for $v_1^{1,2}(x/a)$ should be as follows

$$v_1^{1,2}(x/a) = \mp\frac{\tau_1}{\pi}\alpha_{1,2}\left(\ln|x/a| + \ln(2/\omega_{1,2})\right) + C_{1,2}.$$

Then one comes to a system of linear equations

$$\begin{cases} \left[\left(g^+(0) + g^-(0) + 2\ln\left(\frac{2}{\omega_1}\right)\right)\tau_1 - \frac{2\tau_2}{\pi}\right]\alpha_1 + \\ + \left[(h^+(0) + h^-(0))\,\tau_1 - \frac{2\delta\tau_2}{\pi}\right]\alpha_2 = 0, \\ \left[(h^+(0) + h^-(0))\,\tau_1 - \frac{2\delta\tau_2}{\pi}\right]\alpha_1 + \\ + \left[\left(g^+(0) + g^-(0) + 2\ln(\frac{2}{\omega_2})\right)\tau_1 - \frac{2\tau_2}{\pi}\right]\alpha_2 = 0. \end{cases} \tag{11}$$

Let us introduce the notation

$$A_1 = g^+(0) + g^-(0) + 2\ln(2/\omega_1), \quad A_2 = g^+(0) + g^-(0) + 2\ln(2/\omega_2),$$

$$B = h^+(0) + h^-(0), \quad \beta = \frac{2\tau_2}{\pi}.$$

Then system (11) takes the form

$$\begin{cases} [A_1\tau_1 - \beta]\,\alpha_1 + [B\tau_1 - \delta\beta]\,\alpha_2 = 0, \\ [B\tau_1 - \delta\beta]\,\alpha_1 + [A_2\tau_1 - \beta]\,\alpha_2 = 0. \end{cases} \tag{12}$$

System (12) has a nonzero solution if

$$\begin{vmatrix} A_1\tau_1 - \beta & B\tau_1 - \delta\beta \\ B\tau_1 - \delta\beta & A_2\tau_1 - \beta \end{vmatrix} = 0.$$

For apertures of equal width, one has $\omega_1 = \omega_2$, $A_1 = A_2 = A$. Hence, two values are possible for β:

$$\beta^+ = \frac{A+B}{1+\delta}\tau_1, \quad \beta^- = \frac{A-B}{1-\delta}\tau_1,$$

and

$$\tau_2^+ = \frac{\pi}{2}\frac{A+B}{1+\delta}\tau_1, \quad \tau_2^- = \frac{\pi}{2}\frac{A-B}{1-\delta}\tau_1.$$

Substituting (10), one obtains

$$\tau_2^+ = \frac{\pi^2}{2d_+}\frac{A+B}{(1+\delta)^2}, \quad \tau_2^- = \frac{\pi^2}{2d_+}\frac{A-B}{(1-\delta)^2}.$$

Using $h^{\pm}(0) = g^{\pm}(0) - \frac{1}{\pi}\ln m$, one, finally, comes to the following expressions

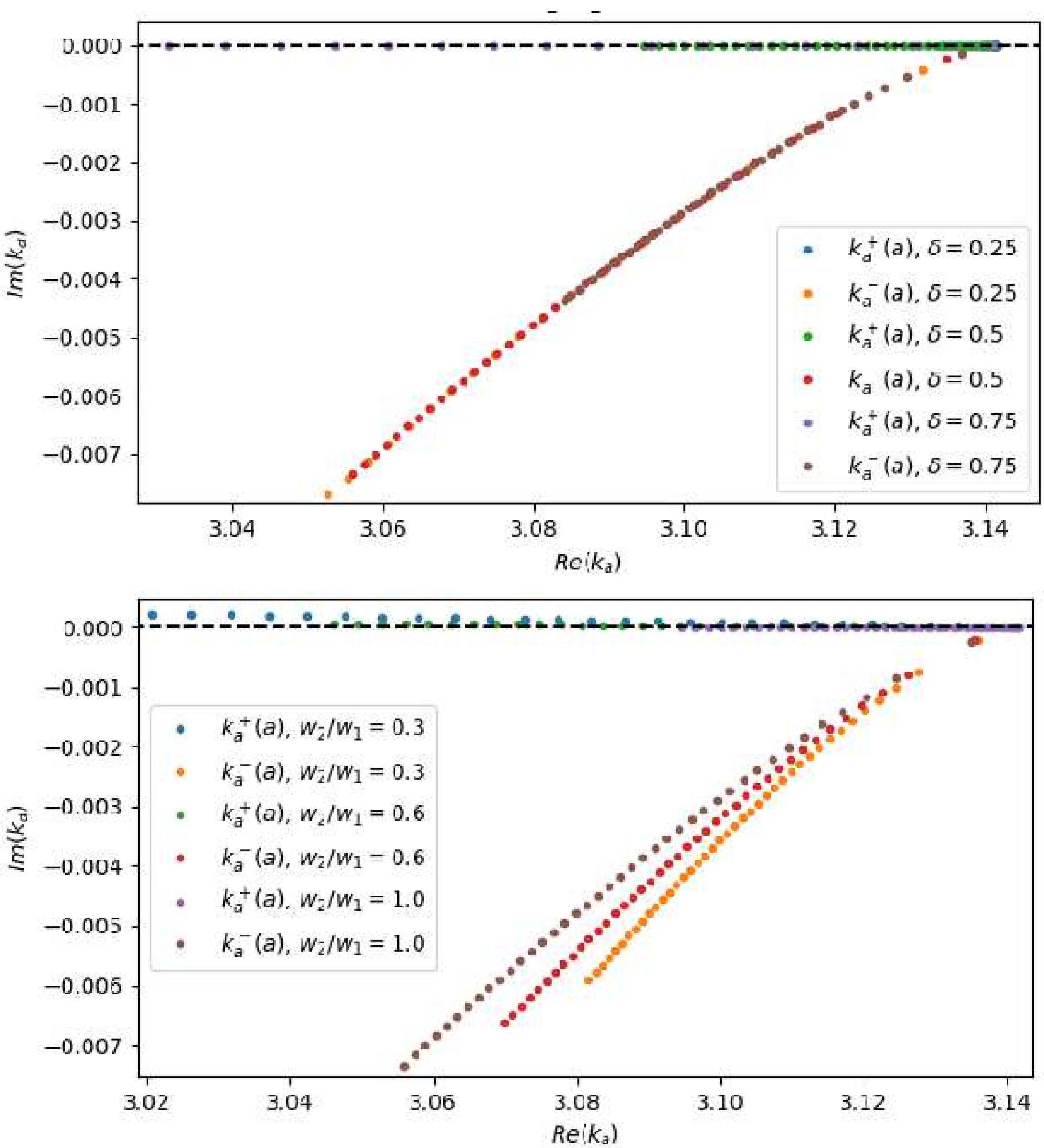
$$\tau_2^+ = \frac{\pi^2}{d_+(1+\delta)^2}\left(g^+(0) + g^-(0) + \ln\frac{2}{\omega} - \frac{1}{\pi}\ln m\right),$$

$$\tau_2^- = \frac{\pi^2}{d_+(1-\delta)^2}\left(\ln\frac{2}{\omega} + \frac{1}{\pi}\ln m\right),$$

where $\omega_1 = \omega_2 = \omega$.

Thus, the proof of the theorem is complete.

4. Discussion



(a)

(b)

Fig. 2. The dependence of the resonance k_a on a for different δ (a) and $\omega 2/\omega 1$ (b)

Fig. 2 (a) represents the behavior of resonance k_a when two equal apertures are decreasing and distance ε between them is decreasing as $\varepsilon = ma^{\delta}$ for different δ. One can see that k_a^+ is real and the speed of convergence δ of the apertures does not affect the shape of the resonance curve.

Fig. 2 (b) shows the resonance curve for the case $\delta = 0.5$ for different values of the ratio of the apertures sizes ω_2/ω_1. If the apertures are not equal the both values of k_a have nonzero imaginary part.

In Fig. 3, the resonance curves for varied ω_2/ω_1 for different values of δ and fixed a are shown.

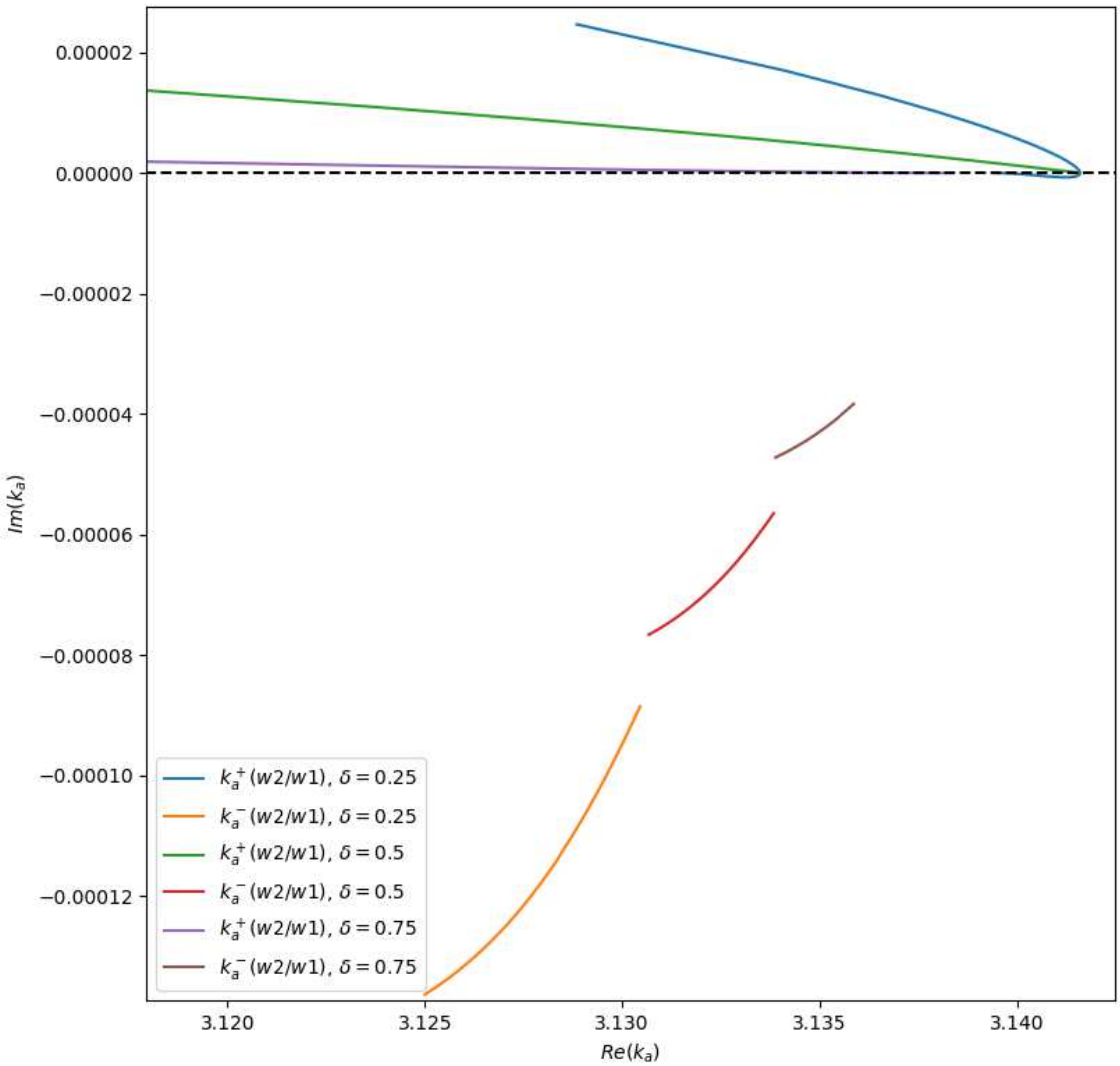


Fig. 3. The dependence of the resonance k_a on ω_2/ω_1 for different δ

The calculations for plotting the graphs (Figs. 2 and 3) are performed numerically and are based on the analytical arguments given in the Appendix.

5. Appendix

Here we propose a way to calculate the remainder $g^+(0)$ of the series for the Green function (5). Namely, (5) gives us:

$$g^+(x) = G^+\Big((x,0);(0,0);k\Big) - \frac{1}{d_+\sqrt{\pi^2/d_+^2 - k^2}} + \frac{1}{\pi}\ln|x| =$$

$$= \frac{e^{-d_+ki|x|}}{2d_+ki} + \frac{1}{\pi}\left[\sum_{n=2}^{\infty} \frac{\exp\Big(-\pi\sqrt{n^2-(kd_+/\pi)^2}|x|\Big)}{\sqrt{n^2-(kd_+/\pi)^2}} - \ln|x|\right].$$

Let $s = kd_+/\pi$ (s is close to 1), $t = \pi|x|$. Then

$$g^+(0) = \frac{1}{2d_+ki} + \frac{1}{\pi}\ln\pi + \frac{1}{\pi}\lim_{t\to 0+}\left[\sum_{n=2}^{\infty}\frac{\exp\Big(-t\sqrt{n^2-s^2}\Big)}{\sqrt{n^2-s^2}} - \ln t\right].$$

To find this limit, one uses the Taylor series for $\ln t$, $0 < t < 2$. Then

$$\sum_{n=2}^{\infty}\frac{\exp\Big(-t\sqrt{n^2-s^2}\Big)}{\sqrt{n^2-s^2}} - \ln t = t - 1 + \sum_{n=2}^{\infty}\left[\frac{\exp\Big(-t\sqrt{n^2-s^2}\Big)}{\sqrt{n^2-s^2}} - \frac{(1-t)^n}{n}\right] =$$
$$= t - 1 + \sum_{n=2}^{\infty}\frac{1}{n}\left[\exp\Big(-t\sqrt{n^2-s^2}\Big) - (1-t)^n\right] +$$
$$+ \sum_{n=2}^{\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1-s^2/n^2}} - 1\right]\exp\Big(-t\sqrt{n^2-s^2}\Big). \quad (13)$$

Obviously, the last series in (13) converges uniformly for $t \in [0, l)$, $l > 0$.

Let us consider the first series in (13) and prove its uniform convergence. Let

$$\varphi_n(x) = \exp\Big(-x\sqrt{n^2-s^2}\Big) - (1-x)^n.$$

We are interested in the upper estimate of the function $\varphi_n(x)$:

$$\max_x \varphi_n = \varphi_n(x_0).$$

If $|x_0(n)| \geqslant C > 0$ for all n, then $\max\varphi_n(x) \leqslant \varphi_n(C)$ and the series $\sum_{n=2}^{\infty}\frac{1}{n}\varphi_n(t)$ converges uniformly (the first series in (13)).

If $x_0(n)$ has a subsequence converging to zero ($\xi_n \to 0$), then

$$\sqrt{1-\frac{s^2}{n^2}}e^{-\xi_n\sqrt{n^2-s^2}} = (1-\xi_n)^{n-1} \quad \Leftrightarrow$$

$$\sqrt{1-\frac{s^2}{n^2}}(1-\xi_n)e^{-A} = 1, \quad A = \xi_n\sqrt{n^2-s^2} + n\ln(1-\xi_n), \quad \Rightarrow$$

$$A \to 0 \quad \Leftrightarrow$$

$$n\ln(1-\xi_n) + n\xi_n \to 0 \quad \Leftrightarrow$$

$$-\frac{1}{2}n\xi_n^2 + R_2(\xi_n)n \to 0,$$

where $R_2(\xi_n)$ is the remainder of the Taylor series for $\ln(1-\xi_n)$. Since $R_2(\xi_n) < 0$, then $\xi_n = o(1/\sqrt{n})$ and using $d\varphi_n/dx(\xi_n) = 0$, one obtains

$$\varphi_n(\xi_n) = (1-\xi_n)^{n-1}\left[\frac{1}{\sqrt{1-s^2/n^2}} - 1 + \xi_n\right] = o\left(\frac{1}{\sqrt{n}}\right),$$

then $\sum_{n=2}^{\infty}\frac{1}{n}\varphi_n(t)$ converges uniformly.

Due to the uniform convergence, the limit (5) gives one

$$\lim_{t\to 0+}\left[\sum_{n=2}^{\infty}\frac{\exp\Big(-t\sqrt{n^2-s^2}\Big)}{\sqrt{n^2-s^2}}-\ln t\right]=-1+\sum_{n=2}^{\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1-s^2/n^2}}-1\right]=$$

$$=-1+\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-s^2}(n+\sqrt{n^2-s^2})}.$$

To calculate this series numerically with an arbitrary predetermined error, we could estimate the remainder:

$$\sum_{n=m+1}^{\infty}\frac{1}{n\sqrt{n^2-s^2}(n+\sqrt{n^2-s^2})}\leqslant\int\limits_{m}^{\infty}\frac{du}{u\sqrt{u^2-s^2}(u+\sqrt{u^2-s^2})}=$$

$$=\ln 2-\ln\Big(1+\sqrt{1-s^2/m^2}\Big).$$

Finally,

$$g^+(0)=\frac{1}{2d_+ki}+\frac{1}{\pi}\ln\pi+\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-s^2}(n+\sqrt{n^2-s^2})},$$

where $s=kd_+/\pi$.

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АСИМПТОТИЧЕСКИЕ РАЗЛОЖЕНИЯ РЕЗОНАНСОВ ДЛЯ ВОЛНОВОДОВ, СВЯЗАННЫХ ЧЕРЕЗ СБЛИЖАЮЩИЕСЯ ОТВЕРСТИЯ

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Рассмотрены двумерные волноводы, связанные через малые отверстия. Получены и исследованы первые члены асимптотических разложений резонансов для случая, когда расстояние между отверстиями уменьшается. Используется метод согласования асимптотических разложений решений краевых задач.

Ключевые слова: *резонанс, асимптотика, связанные волноводы, рассеяние, низкоразмерная система.*

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