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SOME CONGRUENCES INVOLVING INVERSE OF BINOMIAL COEFFICIENTS

L. Khaldi[1,a], **R. Boumahdi**[2,b]

[1] *University of Bouira, Bouira, Algeria*
[2] *University of Science and Technology Houari Boumediene, Bab-Ezzouar, Algeria*
[a] *l.khaldi@univ-bouira.dz*, [b] *r_boumehdi@esi.dz*

Let p be an odd prime number. In this paper, among other results, we establish some congruences involving inverse of binomial coefficients. These congruences are mainly determined modulo p, p^2, p^3 and p^4 in the p-integers ring in terms of Fermat quotients, harmonic numbers and Bernoulli numbers in a simple way. Furthermore, we extend an interesting theorem of E. Lehmer to the class of inverse binomial coefficients.

Keywords: *congruence, binomial coefficient, Fermat quotient, gamma function.*

1. Introduction

For any non-negative integers $n,\ k$ the binomial coefficients are defined by

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & \text{otherwise,} \end{cases}$$

and the harmonic numbers H_n are the rational numbers defined by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}^*.$$

In many mathematics domains, such as combinatorics, graph theory and number theory, binomial coefficients often appear naturally and play an important role. However, it is well-known that it is difficult to compute the values of combinatorial sums involving inverses of binomial coefficients. The gamma function is defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \quad \text{for } x \in \mathbb{R}_+^*,$$

and it is connected to the inverse binomial coefficient via the following relation

$$\frac{1}{\binom{n}{k}} = \frac{k!(n-k)!}{n!} = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+1)} = (n+1) \int_0^1 t^k (1-t)^{n-k} dt. \tag{1}$$

The Bernoulli polynomials $(B_n(x))_{n\geqslant 0}$ may be defined by means of the exponential generating function as follows:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n,$$

and the Bernoulli numbers by $B_n = B_n(0)$, $n \geqslant 0$. The Fermat quotient of a positive integer m with respect to an odd prime p not dividing m is defined by

$$q_m = q_p(m) = \frac{m^{p-1} - 1}{p}. \tag{2}$$

The concept of congruence is very old. Although its origin goes to a distant past, it was not until the eighteenth century, especially with Gauss [1], that it was formulated with a rigorous mathematical language and it was Babbage [2], who in 1819, initiated the congruences for the binomial coefficients by establishing that

$$\binom{2p-1}{p-1} \equiv 1 \left(\operatorname{mod} p^2\right) \quad \text{for } p \geqslant 3. \tag{3}$$

In fact, Babbage thought that he had found a modulo p^2 analogue of the theorem of Wilson concerning the characterization of the prime numbers. So Babbage thought that

$$p \text{ is a prime number if and only if } \binom{2p-1}{p-1} \equiv 1 \left(\operatorname{mod} p^2\right) \text{ for } p \geqslant 3.$$

In fact it is not, since if we consider $N = 283686649 = 16843^2$, we have

$$\binom{2N-1}{N-1} \equiv 1 \left(\operatorname{mod} N^2\right).$$

It has been proven that $N = 283686649$ is the only composed number less than 10^9 satisfying the previous congruence [3]. In 1862, Wolstenholme [4] improved Babbage's results by proving the following two historical congruences

$$H_{p-1} \equiv 0 \left(\operatorname{mod} p^2\right) \text{ for } p \geqslant 5, \tag{4}$$

$$\binom{2p-1}{p-1} \equiv 1 \left(\operatorname{mod} p^3\right) \quad \text{for } p \geqslant 5. \tag{5}$$

The congruence

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \left(\operatorname{mod} p^4\right) \quad \text{for } p \geqslant 5, \tag{6}$$

has been proven in 1900 by Glaisher [5] as an improvement modulo p^4 of Wolstenholme's congruence.

In turn, in 1895, Morley [6] proved that the following congruence holds for any prime $p \geqslant 5$

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} \left(\operatorname{mod} p^3\right).$$

Note that this congruence implies the following one which is easy to prove directly

$$\binom{p-1}{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \left(\operatorname{mod} p\right).$$

This congruence is not a criterion of primality. Ayad and Kihel [7] studied the odd composed numbers n satisfying

$$\binom{n-1}{\frac{n-1}{2}} \equiv (-1)^{\frac{n-1}{2}} \left(\operatorname{mod} n\right). \tag{7}$$

They found that $n = 5907 = 3 \times 11 \times 179$ is the smallest composed integer verifying (7). They conjectured that this congruence does not admit solution n such that n is the product of two prime odd numbers. Morley's congruence was generalized to the following result by Carlitz [8]:

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} + \frac{1}{12}p^3 B_{p-3} \left(\operatorname{mod} p^4\right), \quad \text{for } p \geqslant 5. \tag{8}$$

Congruences concerning binomial coefficients and harmonic numbers have had extensive literature in the last three decades for more details see Z.W. Sun [9] or Meštrović's rich survey [3] and the references therein.

Let us also recall that if p is a prime number, we denote by $\mathbb{Z}_{(p)}$ the set of rational numbers having denominators relatively prime with p. It is easy to prove that $\mathbb{Z}_{(p)}$ is a ring (called p-integers ring). For two reduced rational numbers $\alpha = a/b$ and $\beta = c/d \in \mathbb{Z}_{(p)}$ we write $\alpha \equiv \beta \pmod{p}$, if $ad - bc$ is divisible by p and the denominators b, d are relatively prime with p.

In the present work, we exploit some properties of the inverse binomial coefficients to establish congruences for sums involving these numbers. This paper is organized as follows. Section 2 is dealing with some preliminary lemmas needed in the proofs of the main results. Section 3 is concerned with stating results concerning congruences for

$$\sum_{k=1}^{\frac{p-1}{s}} \frac{1}{\binom{k}{m}}, \qquad s \in \{1, 2\}.$$

Section 4 is devoted to present congruences involving the sums

$$\sum_{k=1}^{\lfloor \frac{p}{r} \rfloor} \frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{r}}{k}}, \qquad r = 2, 3, 4 \text{ and } 6$$

and Section 5 is about further remarks.

2. Auxiliary lemmas

In this section, we state some basic facts which will be used in the proofs of our theorems.

Lemma 1. *Let p be an odd prime, then*

$$\binom{p-1}{m-1} \equiv (-1)^{m-1}(1 - pH_{m-1}) \left(\operatorname{mod} p^2\right), \tag{9}$$

holds true for each $m = 1, 2, \ldots, p$.

Proof. We have

$$(-1)^{m-1}\binom{p-1}{m-1} = \prod_{i=1}^{m-1}\left(1 - \frac{p}{i}\right) \equiv 1 - pH_{m-1} \left(\operatorname{mod} p^2\right),$$

which immediately gives the congruence (9). □

Lemma 2. *For any positive integers n, m with $m \leqslant n$ we have*

$$\sum_{k=m}^{n} \frac{1}{\binom{k}{m}} = \begin{cases} H_n, & \text{if } m = 1; \\ \frac{m}{m-1}\left(1 - \frac{1}{\binom{n}{m-1}}\right), & \text{if } m \neq 1. \end{cases} \tag{10}$$

Proof. If $m = 1$, it is clear that

$$\sum_{k=1}^{n} \frac{1}{\binom{k}{1}} = H_n.$$

If $m \neq 1$, in one hand, using the identities

$$\binom{k+1}{m} - \binom{k}{m} = \binom{k}{m-1} \text{ and } \binom{k}{m} = \frac{k}{m}\binom{k-1}{m-1},$$

we obtain

$$\frac{m}{m-1}\left(\frac{1}{\binom{k-1}{m-1}} - \frac{1}{\binom{k}{m-1}}\right) = \frac{m}{m-1}\frac{\binom{k}{m-1} - \binom{k-1}{m-1}}{\binom{k-1}{m-1}\binom{k}{m-1}} = \frac{m}{m-1}\frac{\binom{k-1}{m-2}}{\binom{k-1}{m-1}\binom{k}{m-1}} =$$

$$= \frac{m}{m-1}\frac{\binom{k-1}{m-2}}{\binom{k-1}{m-1}\left(\frac{k}{m-1}\binom{k-1}{m-2}\right)} = \frac{1}{\frac{k}{m}\binom{k-1}{m-1}} = \frac{1}{\binom{k}{m}}.$$

In the other hand, let f be an arbitrary real function, we define the difference operator by $\nabla f(x) = f(x+1) - f(x)$. For $f(t) = \frac{1}{\binom{t}{m-1}}$ we have

$$\sum_{k=m}^{n} \frac{1}{\binom{k}{m}} = \frac{m}{m-1}\sum_{k=m}^{n}\left(\frac{1}{\binom{k-1}{m-1}} - \frac{1}{\binom{k}{m-1}}\right) =$$

$$= -\frac{m}{m-1}\sum_{k=m}^{n} \nabla f(k) = -\frac{m}{m-1}(f(n) - f(m-1)) = \frac{m}{m-1}\left(1 - \frac{1}{\binom{n}{m-1}}\right).$$

Which gives the desired result. □

Lemma 3. *For any odd prime integer p and any $m \in \{1, \ldots, \frac{p+1}{2}\}$ we have*

$$\binom{\frac{p-1}{2}}{m-1} \equiv \frac{(-1)^{m-1}}{2^{2m-2}}\binom{2m-2}{m-1}\left(1 - p\sum_{i=1}^{m-1}\frac{1}{2i-1}\right) \pmod{p^2}.$$

In particular,

$$\binom{\frac{p-1}{2}}{m-1} \equiv \frac{(-1)^{m-1}}{2^{2m-2}}\binom{2m-2}{m-1} \pmod{p}.$$

Proof. For $m \in \{1, 2, \ldots, \frac{p+1}{2}\}$, the definition of the binomial coefficient yields to

$$\binom{\frac{p-1}{2}}{m-1} = \frac{1}{(m-1)!}\frac{p-1}{2}\left(\frac{p-1}{2} - 1\right)\cdots\left(\frac{p-1}{2} - (m-1) + 1\right) =$$

$$= \frac{1}{2^{m-1}(m-1)!}(p-1)(p-3)(p-5)\cdots(p - (2(m-1) - 1)) =$$

$$= \frac{(-1)^{m-1}}{2^{m-1}(m-1)!}\prod_{k=1}^{m-1}(2k-1)\prod_{k=1}^{m-1}\left(1 - \frac{p}{2k-1}\right) = \frac{(-1)^{m-1}}{2^{2m-2}}\binom{2m-2}{m-1}\prod_{k=1}^{m-1}\left(1 - \frac{p}{2k-1}\right).$$

Therefore it easy to show that

$$\prod_{k=1}^{m-1}\left(1-\frac{p}{2k-1}\right)\equiv\left(1-p\sum_{i=1}^{m-1}\frac{1}{2i-1}\right).$$

Then

$$\binom{\frac{p-1}{2}}{m-1}\equiv\frac{(-1)^{m-1}}{2^{2m-2}}\binom{2m-2}{m-2}\left(1-p\sum_{i=1}^{m-1}\frac{1}{2i-1}\right)\left(\operatorname{mod}p^2\right).$$

□

Lemma 4. [10]. *Let p be a prime such that $4p=a^2+3b^2\equiv 1\,(\operatorname{mod}3)$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\,(\operatorname{mod}3)$. Then*

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}\equiv -a+\frac{p}{a}\left(\operatorname{mod}p^2\right). \tag{11}$$

Lemma 5. *For any positive integer n we have*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{\binom{2n}{k}}=\frac{1}{2n+2}+\frac{(-1)^{n-1}}{2\binom{2n}{n}}. \tag{12}$$

In particular,

$$\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{\binom{2n}{k}}=\frac{1}{2n+2}+\frac{(-1)^{n}}{4\binom{2n-1}{n-1}}. \tag{13}$$

Proof. Let us consider $I_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{\binom{2n}{k}}$, then using the relation (1) relating gamma function and binomial coefficient one obtain

$$\begin{aligned}I_n&=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{\binom{2n}{k}}=\sum_{k=1}^{n}(-1)^{k-1}(2n+1)\int_0^1 t^k(1-t)^{2n-k}dt=\\&=-(2n+1)\int_0^1(1-t)^{2n}\left(\sum_{k=1}^{n}\left(\frac{-t}{1-t}\right)^k\right)dt,\end{aligned}$$

therefore by the identity $\sum_{k=1}^{n}x^k=\frac{x(x^n-1)}{x-1}$ and after simplifying, it follows

$$\begin{aligned}I_n&=(-1)^{n-1}(2n+1)\int_0^1 t^{n+1}(1-t)^n dt+(2n+1)\int_0^1 t(1-t)^{2n}dt=\\&=(-1)^{n-1}\frac{2n+1}{2n+2}\frac{1}{\binom{2n+1}{n+1}}+\frac{2n+1}{2n+2}\frac{1}{\binom{2n+1}{1}}=\frac{(-1)^{n-1}}{2\binom{2n}{n}}+\frac{1}{2n+2}.\end{aligned}$$

□

Note that identities (10) and (12) can be found without proof in [11].

Lemma 6. [12]. *Let p be a prime of the form $4q+1$ such that $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\,(\operatorname{mod}4)$. Then*

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv\left(1+\frac{1}{2}pq_2\right)\left(2a-\frac{p}{2a}\right)\left(\operatorname{mod}p^2\right). \tag{14}$$

Lemma 7. [13]. *Let be a prime such that* $p \equiv 1 \pmod 6$ *and* $p = a^2 + 3b^2$ *with* $a, b \in \mathbb{Z}$ *and* $a \equiv -1 \pmod 3$. *Then*

$$\binom{\frac{p-1}{3}}{\frac{p-1}{6}} \equiv \begin{cases} 2(-1)^{\frac{p-1}{6}+1} a \pmod p, & \text{if } b \equiv 0 \pmod 3; \\ (-1)^{\frac{p-1}{6}}(a+3b) \pmod p, & \text{if } b \equiv 1 \pmod 3; \\ (-1)^{\frac{p-1}{6}}(a-3b) \pmod p, & \text{if } b \equiv 2 \pmod 3. \end{cases}$$

Lemma 8. [10]. *Let* p *be a prime of the form* $6q+1$ *such that* $4p = a^2 + 3b^2$ *with* $a, b \in \mathbb{Z}$ *and* $a \equiv 1 \pmod 3$. *Then*

$$\binom{\frac{p-1}{3}}{\frac{p-1}{6}} \equiv (-1)^{\frac{p-1}{6}+1} \left(a - \frac{p}{a}\right)\left(1 + \frac{2^p - 2}{3}\right) \pmod{p^2}.$$

3. Congruences for sums of the inverse of binomial coefficients

In this section we determine $\sum_{k=m}^{\frac{p-1}{s}} \frac{1}{\binom{k}{m}}$ modulo p and p^2 for $s \in \{1, 2\}$ in terms of harmonic numbers, inverse binomial coefficients and Fermat quotient.

Theorem 5. *Let* $p \geqslant 5$ *be a prime and let* $m \in \{1, 2, \ldots, p\}$. *Then we have*

$$\sum_{k=m}^{p-1} \frac{1}{\binom{k}{m}} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } m = 1; \\ \frac{m}{m-1}(1 - (-1)^{m-1}(1 + pH_{m-1})) \pmod{p^2}, & \text{if } m \neq 1. \end{cases}$$

Proof. Taking $n = p-1$ in the congruence (10) of Lemma 2, we get

$$\sum_{k=m}^{p-1} \frac{1}{\binom{k}{m}} = \begin{cases} H_{p-1}, & \text{if } m = 1; \\ \frac{m}{m-1}\left(1 - \frac{1}{\binom{p-1}{m-1}}\right), & \text{if } m \neq 1. \end{cases}$$

In virtue of the congruence (9) of Lemma 1 and using the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \tag{15}$$

which is valid for any $x \in p\mathbb{Z}_{(p)}$, and also by the congruence (4), we conclude

$$\sum_{k=m}^{p-1} \frac{1}{\binom{k}{m}} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } m = 1; \\ \frac{m}{m-1}(1 - (-1)^{m-1}(1 + pH_{m-1})) \pmod{p^2}, & \text{if } m \neq 1. \end{cases}$$

□

Corollary 1. *Let* $p \geqslant 5$ *be a prime and let* $m \in \{1, 2, \ldots, p\}$, *we have*

$$\sum_{k=m}^{p-1} \frac{1}{\binom{k}{m}} \equiv \begin{cases} 0 \pmod p, & \text{if } m = 1; \\ \frac{m}{m-1}(1 + (-1)^m) \pmod p, & \text{if } m \neq 1. \end{cases} \tag{16}$$

Note that when $p \geqslant 5$ is a prime and $m \in \{1, 2, \ldots, p\}$, we also have

$$\sum_{k=m}^{p-1} \frac{1}{\binom{k}{m}} \equiv \begin{cases} 0 \pmod p, & \text{if } m = 1; \\ \frac{2m}{m-1} \pmod p, & \text{if } m \neq 1 \ m \text{ is even}; \\ 0 \pmod p, & \text{if } m \neq 1 \ m \text{ is odd}. \end{cases}$$

Theorem 7. *Let $p \neq 3$ be an odd prime number and let $m \in \{1, \dots, \frac{p-1}{2}\}$. Then*

$$\sum_{k=m}^{\frac{p-1}{2}} \frac{1}{\binom{k}{m}} \equiv \begin{cases} -2q_2 + pq_2^2 \pmod{p^2}, & \text{if } m = 1; \\ \frac{m}{m-1}\left(1 - (-4)^{m-1}\binom{2m-2}{m-1}^{-1}\left(1 + p\sum_{i=1}^{m-1}\frac{1}{2i-1}\right)\right) \pmod{p^2}, & \text{if } m \neq 1, \end{cases} \tag{17}$$

Proof. Taking $n = \frac{p-1}{2}$ in the identity (10) of Lemma 2, we obtain

$$\sum_{k=m}^{\frac{p-1}{2}} \frac{1}{\binom{k}{m}} = \begin{cases} H_{\frac{p-1}{2}}, & \text{if } m = 1; \\ \frac{m}{m-1}\left(1 - \frac{1}{\binom{\frac{p-1}{2}}{m-1}}\right), & \text{if } m \neq 1. \end{cases} \tag{18}$$

If $m = 1$, by the congruence $\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \equiv -2q_2 + pq_2^2 \pmod{p^2}$ (see [14, Congruence (45)]) we deduce the first congruence of (17). If $m \neq 1$, from Lemma 3 and identity (15), we get

$$\begin{aligned} \frac{1}{\binom{\frac{p-1}{2}}{m-1}} &\equiv (-1)^{m-1}4^{1-m}\binom{2m-2}{m-1}^{-1}\frac{1}{1 - p\sum_{i=1}^{m-1}\frac{1}{2i-1}} \equiv \\ &\equiv (-1)^{m-1}4^{1-m}\binom{2m-2}{m-1}^{-1}\left(1 + p\sum_{i=1}^{m-1}\frac{1}{2i-1}\right) \pmod{p^2}. \end{aligned} \tag{19}$$

Substituting (19) in the relation (18) and after simplifying, we deduce the second part of the congruence (17). □

The reduction modulo p of the congruence (17) gives the following corollary.

Corollary 2. *Let $p \neq 3$ be an odd prime number and let $m \in \{1, \dots, \frac{p-1}{2}\}$. Then*

$$\sum_{k=m}^{\frac{p-1}{2}} \frac{1}{\binom{k}{m}} \equiv \begin{cases} -2q_2 \pmod{p}, & \text{if } m = 1; \\ \frac{m}{m-1}\left(1 - (-4)^{m-1}\binom{2m-2}{m-1}^{-1}\right) \pmod{p}, & \text{if } m \neq 1. \end{cases} \tag{20}$$

It should be noted that when $m = 1$, the congruence (17) is equivalent to the congruence mentioned above [14, Congruence (45)] and if $m = 1$, the congruence (20) can be found in [15].

4. Congruences concerning Fermat quotient

This section is devoted to present the congruences $\sum_{k=1}^{\lfloor \frac{p}{r} \rfloor} \frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{r}}{k}}$, $r = 2, 3, 4, 6$, modulo p, p^2 and p^4 in terms of Fermat quotient and Bernoulli numbers.

Theorem 9. *Let $p \geqslant 5$ be a prime, we have*

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^{k-1}}{\binom{p-1}{k}} \equiv \frac{1}{2} - p(1 - q_2) + p^2\left(1 - \frac{3}{2}q_2^2\right) - p^3\left(1 - 2q_2^3 - \frac{1}{24}B_{p-3}\right) \pmod{p^4}.$$

Proof. Taking $n=\frac{p-1}{2}$ in the identity (12) of Lemma 5, we write

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^{k-1}}{\binom{p-1}{k}}=\frac{1}{1+p}-\frac{(-1)^{\frac{p-1}{2}}}{2\binom{p-1}{\frac{p-1}{2}}}. \tag{21}$$

By the identity (15), we have

$$\frac{1}{1+p}\equiv 1-p+p^2-p^3\left(\operatorname{mod}p^4\right). \tag{22}$$

Thereafter by the congruence (8), we obtain

$$\frac{(-1)^{\frac{p-1}{2}}}{\binom{p-1}{\frac{p-1}{2}}}\equiv\frac{1}{4^{p-1}+\frac{1}{12}p^3B_{p-3}}\left(\operatorname{mod}p^4\right). \tag{23}$$

The equality (2) which define the Fermat quotient implies $4^{p-1}=(2^{p-1})^2=1+2pq_2+p^2q_2^2$, replacing in (23) with identity (15), we obtain

$$\frac{(-1)^{\frac{p-1}{2}}}{\binom{p-1}{\frac{p-1}{2}}}\equiv\frac{1}{1+2pq_2+p^2q_2^2+\frac{1}{12}p^3B_{p-3}}\left(\operatorname{mod}p^4\right)\equiv$$

$$\equiv 1-2pq_2+3p^2q_2^2-4p^3q_2^3-\frac{1}{12}p^3B_{p-3}\left(\operatorname{mod}p^4\right). \tag{24}$$

Combining the congruences (22), (24) and identity (21), it comes

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^{k-1}}{\binom{p-1}{k}}\equiv\frac{1}{2}-p(1-q_2)+p^2\left(1-\frac{3}{2}q_2^2\right)-p^3\left(1-2q_2^3-\frac{1}{24}B_{p-3}\right)\left(\operatorname{mod}p^4\right).$$

□

Theorem 11. *Let p be a prime such that $4p=a^2+3b^2\equiv 1\,(\operatorname{mod}3)$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\,(\operatorname{mod}3)$. Then*

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}\frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{3}}{k}}\equiv\frac{3}{4}\left(1-\frac{1}{2}p\right)+\frac{(-1)^{\frac{p-1}{3}}}{2a}\left(1+\frac{p}{a^2}\right)\left(\operatorname{mod}p^2\right).$$

In particular,

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}\frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{3}}{k}}\equiv\frac{3}{4}+\frac{(-1)^{\frac{p-1}{3}}}{2a}\,(\operatorname{mod}p).$$

Proof. Taking $n=\frac{p-1}{3}$ in the identity (12) of Lemma 5, we obtain

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}\frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{3}}{k}}=\frac{3}{4}\frac{1}{1+\frac{1}{2}p}-\frac{(-1)^{\frac{p-1}{3}}}{2\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}}.$$

In view of the congruence (11) of Lemma 4, we write

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}\frac{(-1)^{k-1}}{\binom{\frac{2(p-1)}{3}}{k}}\equiv\frac{3}{4}\frac{1}{1+\frac{1}{2}p}-\frac{(-1)^{\frac{p-1}{3}}}{2\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}}\left(\operatorname{mod}p^2\right)\equiv$$

$$\equiv \frac{3}{4}\left(1-\frac{1}{2}p\right)-\frac{(-1)^{\frac{p-1}{3}}}{2}\frac{1}{-a+\frac{p}{a}} \,(\mathrm{mod}\, p^2) \equiv \frac{3}{4}\left(1-\frac{1}{2}p\right)+\frac{(-1)^{\frac{p-1}{3}}}{2a}\left(1+\frac{p}{a^2}\right) (\mathrm{mod}\, p^2).$$

□

Theorem 13. *Let p a be prime of the form $4q+1$ and $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\,(\mathrm{mod}\, 4)$. Then*

$$\sum_{k=1}^{\lfloor \frac{p}{4}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{2}}{k}} \equiv \frac{2}{3}\left(1-\frac{1}{3}p\right) - \frac{(-1)^{\frac{p-1}{4}}}{4a}\left(1+\frac{1}{4a^2}p-\frac{q_2}{2}p\right) (\mathrm{mod}\, p^2).$$

Proof. Taking $n=\frac{p-1}{4}$ in identity (12) of Lemma 5, we get

$$\sum_{k=1}^{\lfloor \frac{p}{4}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{2}}{k}} = \frac{2}{3}\frac{1}{1+\frac{1}{3}p} - \frac{(-1)^{\frac{p-1}{4}}}{2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}},$$

then using the congruence (14) of Lemma 6, we have

$$\sum_{k=1}^{\lfloor \frac{p}{4}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{2}}{k}} \equiv \frac{2}{3}\left(1-\frac{1}{3}p\right) - \frac{(-1)^{\frac{p-1}{4}}}{2(1+\frac{1}{2}pq_2)(2a-\frac{p}{2a})} (\mathrm{mod}\, p^2) \equiv$$

$$\equiv \frac{2}{3}\left(1-\frac{1}{3}p\right) - \frac{(-1)^{\frac{p-1}{4}}}{4a}\left(1+\frac{1}{4a^2}p-\frac{q_2}{2}p\right) (\mathrm{mod}\, p^2).$$

□

Corollary 3. *Let p be a prime of the form $4q+1$ and $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\,(\mathrm{mod}\, 4)$. Then*

$$\sum_{k=1}^{\lfloor \frac{p}{4}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{2}}{k}} \equiv \frac{2}{3} - \frac{(-1)^{\frac{p-1}{4}}}{4a} (\mathrm{mod}\, p).$$

Theorem 15. *Let $p\equiv 1\,(\mathrm{mod}\, 6)$ be a prime and $p=a^2+3b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv -1\,(\mathrm{mod}\, 3)$. Then*

$$\sum_{k=1}^{\lfloor \frac{p}{6}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} \equiv \begin{cases} \frac{3}{5}+\frac{1}{4a}\,(\mathrm{mod}\, p), & \textit{if } b\equiv 0\,(\mathrm{mod}\, 3);\\ \frac{3}{5}-\frac{1}{2(a+3b)}\,(\mathrm{mod}\, p), & \textit{if } b\equiv 1\,(\mathrm{mod}\, 3);\\ \frac{3}{5}-\frac{1}{2(a-3b)}\,(\mathrm{mod}\, p), & \textit{if } b\equiv 2\,(\mathrm{mod}\, 3).\end{cases}$$

Proof. Taking $n=\frac{p-1}{6}$ in the identity (12) of Lemma 5, we have

$$\sum_{k=1}^{\lfloor \frac{p}{6}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} = \frac{3}{5}\frac{1}{1+\frac{1}{5}p} - \frac{(-1)^{\frac{p-1}{6}}}{2\binom{\frac{p-1}{3}}{\frac{p-1}{6}}}.$$

In virtue of Lemma 7, we get

$$\sum_{k=1}^{\lfloor \frac{p}{6}\rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} \equiv \begin{cases} \frac{3}{5}+\frac{1}{4a}\,(\mathrm{mod}\, p), & \text{if } b\equiv 0\,(\mathrm{mod}\, 3);\\ \frac{3}{5}-\frac{1}{2(a+3b)}\,(\mathrm{mod}\, p), & \text{if } b\equiv 1\,(\mathrm{mod}\, 3);\\ \frac{3}{5}-\frac{1}{2(a-3b)}\,(\mathrm{mod}\, p), & \text{if } b\equiv 2\,(\mathrm{mod}\, 3).\end{cases}$$

□

Theorem 17. *Let p be a prime of the form $6q+1$ such that $4p = a^2 + 3b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 3$. Then*

$$\sum_{k=1}^{\lfloor \frac{p}{6} \rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} \equiv \frac{3}{5} + \frac{1}{2a} - p\left(\frac{3}{25} - \frac{1}{2a^3} + \frac{1}{3a} q_2\right) \pmod{p^2}.$$

Proof. Taking $n = \frac{p-1}{6}$ in the identity (12) of Lemma 5, it comes

$$\sum_{k=1}^{\lfloor \frac{p}{6} \rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} = \frac{3}{5} \frac{1}{1 + \frac{1}{5}p} - \frac{(-1)^{\frac{p-1}{6}}}{2\binom{\frac{p-1}{3}}{\frac{p-1}{6}}}.$$

Thanks to Lemma 8, we get

$$\begin{aligned}\sum_{k=1}^{\lfloor \frac{p}{6} \rfloor} \frac{(-1)^{k-1}}{\binom{\frac{p-1}{3}}{k}} &\equiv \frac{3}{5}\left(1 - \frac{1}{5}p\right) + \frac{1}{2(a - \frac{p}{a})(1 + \frac{2^p - 2}{3})} \pmod{p^2} \equiv \\ &\equiv \frac{3}{5}\left(1 - \frac{1}{5}p\right) + \frac{1}{2a(1 - \frac{p}{a^2})(1 + \frac{2}{3}pq_2)} \pmod{p^2} \equiv \\ &\equiv \frac{3}{5} + \frac{1}{2a} - p\left(\frac{3}{25} - \frac{1}{2a^3} + \frac{1}{3a} q_2\right) \pmod{p^2}.\end{aligned}$$

□

5. Remarks

Remark 1. Recall that the falling factorial $z^{\underline{m}}$ is the polynomial in z defined by $z^{\underline{0}} := 1$ and $z^{\underline{m}} := \prod_{j=0}^{m-1} (z - j)$ for any integer $m \geqslant 1$. Since $\binom{k}{m} = \frac{k^{\underline{m}}}{m!}$, we easily may rewrite the congruence (16) in the form

$$\sum_{k=m}^{p-1} \frac{1}{k^{\underline{m}}} \equiv \begin{cases} 0 \pmod p, & \text{if } m = 1; \\ \frac{1 + (-1)^m}{(m-2)!(m-1)^2} \pmod p, & \text{if } m \neq 1. \end{cases}$$

Remark 2. If we consider the case of infinite power sums involving inverse of binomial coefficients

$$\zeta_m(n) = \sum_{k=m}^{\infty} \frac{1}{\binom{k}{m}^n}, \quad \text{for } n,\ m \geqslant 1,$$

then by taking $m = 1$, we have

$$\zeta(n) = \zeta_1(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad \text{for } n \geqslant 2,$$

where $\zeta(n)$ is the Riemann zeta function. The special case for $n = 1$ and by Lemma 2, we obtain

$$\zeta_m(1) = \sum_{k=m}^{\infty} \frac{1}{\binom{k}{m}} = \frac{m}{m-1}, \text{ for } m \geqslant 2.$$

For any integer $n \geqslant 1$, then

$$\zeta(2n) = -\frac{1}{2}\frac{(2i\pi)^{2n}}{(2n)!}B_{2n}, \ n = 1, 2, \ldots$$

So we can calculate $\zeta_m(n)$ according to $\zeta(n)$ for example

$$\zeta_2(2) = \sum_{k=2}^{\infty} \frac{1}{\binom{k}{2}^2} = -12 + 8\zeta(2) \text{ and } \zeta_2(3) = \sum_{k=2}^{\infty} \frac{1}{\binom{k}{2}^3} = 80 - 48\zeta(2).$$

Remark 3. Setting $n = p$ in the identity (13), and by using the congruence (3) of Babbage we obtain

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{\binom{2p}{k}} \equiv \frac{1}{2}\frac{1}{1+p} - \frac{1}{4}\frac{1}{\binom{2p-1}{p-1}} \equiv \frac{1}{4} - \frac{1}{2}p \left(\operatorname{mod} p^2\right), \text{ for } p \geqslant 3.$$

From congruence (5) we deduce

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{\binom{2p}{k}} \equiv \frac{1}{4} - \frac{1}{2}p + \frac{1}{2}p^2 \left(\operatorname{mod} p^3\right), \text{ for } p \geqslant 5.$$

Then (6) yields to

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{\binom{2p}{k}} \equiv \frac{1}{4} - \frac{1}{2}p + \frac{1}{2}p^2 - \left(\frac{1}{2} + \frac{1}{6}B_{p-3}\right) p^3 \left(\operatorname{mod} p^4\right), \text{ for } p \geqslant 3.$$

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НЕКОТОРЫЕ СООТВЕТСТВИЯ, ВКЛЮЧАЮЩИЕ ОБРАТНЫЕ БИНОМИАЛЬНЫЕ КОЭФФИЦИЕНТЫ

Л. Халди, Р. Боумахди

[1] *Университет Буйры, Буйра, Алжир*

[2] *Университет науки и технологии Хуари Бумедьен, Баб-Эззуар, Алжир*

[a] *l.khaldi@univ-bouira.dz,* [b] *r_boumehdi@esi.dz*

Пусть p — нечётное простое число. В этой статье, среди прочих результатов, мы доказываем некоторые соответствия, включающие обратные биномиальные коэффициенты. Эти соответствия в основном определяются по модулю p, p^2, p^3 и p^4 в кольце p-целых чисел в терминах коэффициентов Ферма, гармонических чисел и чисел Бернулли простым способом. Кроме того, мы распространяем интересную теорему Э. Лемера на класс обратных биномиальных коэффициентов.

Ключевые слова: *конгруэнтность, биномиальный коэффициент, коэффициент Ферма, гамма-функция.*

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Сведения об авторах

Халди Лаала, ассистент лаборатории компьютерных наук, математики и физики для сельского и лесного хозяйства, кафедра математики, Университет Буйры, Буйра, Алжир; e-mail: l.khaldi@univ-bouira.dz.

Боумахди Рашид, доцент Национальной высшей школы математики и лаборатории арифметики, кодирования, комбинаторики и криптографии, кафедра алгебры и теории чисел, Университет науки и технологии Хуари Бумедьен, Баб-Эззуар, Алжир; e-mail: hml@ciencias.unam.mx.