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ON STEPANOV-LIKE ALMOST PERIODICITY IN MIXED LEBESGUE SPACES

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The main purpose of this paper is to revisit the recently analyzed class of multi-dimensional Stepanov almost periodic functions. We introduce and study several new classes of Stepanov-like almost periodic functions in the mixed Lebesgue spaces. We also provide a new application of multi-dimensional Stepanov almost periodic functions to the abstract nonautonomous differential equations of first order, provided that all components of the exponent $\vec{p} \in [1, \infty)^n$ are equal.

Keywords: *Stepanov-like almost periodic function in mixed Lebesgue space, multi-dimensional Stepanov almost periodic function, abstract nonautonomous differential equation of first order.*

1. Introduction and preliminaries

The notion of almost periodicity, introduced by H. Bohr [1] around 1924–1926, is an attractive topic in the qualitative theory of differential equations. Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $(X, \| \cdot \|_X)$ be a complex Banach space. For a given $\varepsilon > 0$, we call $\tau > 0$ a ε-period for $f : I \to X$ if and only if $\|f(t + \tau) - f(t)\|_X \leqslant \varepsilon$, $t \in I$. The set constituted of all ε-periods for f is denoted by $\vartheta(f, \varepsilon)$. It is said that f is almost periodic if and only if for every $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \varepsilon)$.

Without any substantial difficulty, the notion of almost periodicity can be extended to the multi-dimensional case (for more details, see the article [2], the research monograph [3] and the list of references cited therein; let us also mention here that the almost periodic functions on locally compact abelian groups have been considered in the research monograph [4] by Y. Katznelson). Let $\emptyset \neq I \subseteq \mathbb{R}^n$, $t_0 \in \mathbb{R}^n$ and $l > 0$. Set$B_I(t_0, l) := \{t \in I : |t - t_0| \leqslant l\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Then we say that a continuous function $F : I \to X$ is almost periodic if and only if for each $\varepsilon > 0$ there exists $l > 0$ such that for each $t_0 \in I$ there exists $\tau \in B_I(t_0, l)$ such that$\|F(t + \tau) - F(t)\|_X \leqslant \varepsilon$, $t \in I$. The vector space consisting of all almost periodic functions on I is denoted by $AP(I : X)$.

The almost periodic solutions for different classes of ordinary differential equations and partial differential equations, modeling various real world phenomena in physics, mathematical biology, control theory, crystallography, etc., have been studied by many authors (see e.g., the research monographs [5–12] for more details about the subject). The various notions of almost periodicity in the multi-dimensional setting have recently

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been introduced and studied by A. Chávez, K. Khalil, M. Kostić and M. Pinto in [2] (see also the research articles [13] by V.E. Fedorov, M. Kostić, [14] by D. Lenz, T. Spindeler, N. Strungaru, [15] by T. Spindeler and the research monograph [10] by A.A. Pankov for more details about the Weyl and Besicovitch classes of multi-dimensional almost periodic functions). As is well known, the Stepanov p-almost periodicity is a significant extension of the classical Bohr almost periodicity from continuous functions to the locally Lebesgue p-integrable functions ($1 \leqslant p < \infty$). The main aim of this paper is to reconsider our recent results from [16] by examining the Stepanov-like almost periodic functions in mixed Lebesgue spaces.

The organization of paper can be briefly described as follows. In Section 2, we recall the basic notions and results about the mixed Lebesgue spaces we are working with. Our main results about Stepanov-like almost periodic functions in mixed Lebesgue spaces are presented in Section 3. Subsection 3.1 investigates the Stepanov $\vec{p}$-bounded functions depending on two parameters, whilest Subsection 3.2 investigates several various classes of Stepanov-like almost periodic functions in mixed Lebesgue spaces; in Subsection 3.2, we actually report how our recent results from [16] can be restated in this new context ($\vec{p} \in [1, \infty)^n$). We also present an interesting application to the abstract nonautonomous first-order differential equations at the end of paper.

We use the standard terminology throughout the paper. We assume that Λ is a non-empty subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ as well as that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two complex Banach spaces. By $\mathcal{B}$ we denote a non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. If A and B are non-empty sets, then we define $B^A \equiv \{f \mid f : A \to B\}$.

Let $1 \leqslant p < \infty$ and $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$. By $L^p(\Lambda : X)$ we denote the vector space of Lebesgue measurable functions $F : \Lambda \to X$ such that

$$\int_{\Lambda} \|F(t)\|_X^p \, dt < \infty;$$

$L^\infty(\Lambda : X)$ is defined as the vector space consisting of all Lebesgue measurable functions that are bounded almost everywhere on Λ.

Set

$$\|F\|_{L^p} := \int_{\Lambda} \|F(t)\|_X^p \, dt \text{ for } 1 \leqslant p < \infty;$$
$$\|F\|_{L^\infty} := \inf \{c > 0 : \|F(t)\|_X \leqslant c \text{ almost everywhere on } \Lambda\} \text{ for } p = +\infty.$$

We know that $(L^p(\Lambda : X), \|\cdot\|_{L^p})$ is a Banach space ($1 \leqslant p \leqslant +\infty$).

In what follows, we recall some basic definitions and results about the Stepanov p-almost periodic functions over the interval $I = \mathbb{R}$ or $I = [0, \infty)$.

Definition 1. Let $1 \leqslant p < \infty$. A Lebesgue measurable function $f : I \to X$ is said to be *Stepanov p-bounded* if and only if

$$\|f\|_{S^p} := \sup_{t \in I} \left(\int_0^1 \|f(t+s)\|_X^p \, ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $BS^p(I : X)$ consisted of all Stepanov p-bounded functions is a Banach space.

A function $f \in L^p_{loc}(I : X)$, where $1 \leqslant p < \infty$, is said to be Stepanov p-almost periodic if and only if its Bochner transform $f^b : I \to L^p([0,1] : X)$, defined by

$f^b(t)(s) := f(t+s)$, $t \in I$, $s \in [0,1]$, is almost periodic, i. e., for every $\varepsilon > 0$, there is a real number $l > 0$ such that any subinterval $I' \subseteq I$ of length l contains at least one point $\tau \in \mathbb{R} \setminus \{0\}$ such that

$$\sup_{t \in I} \left(\int_0^1 \|f(t+s+\tau) - f(t+s)\|_X^p \, ds \right)^{1/p} \leqslant \varepsilon.$$

By $S^p_{AP}(I : X)$ we denote the set consisting of all Stepanov p-almost periodic functions $f : I \to \mathbb{R}$.

For $1 \leqslant q \leqslant p < \infty$, we have the following continuous embeddings

$$S^p_{AP}(I : X) \hookrightarrow S^q_{AP}(I : X) \hookrightarrow S^1_{AP}(I : X),$$

and

$$AP(I : X) \hookrightarrow S^p_{AP}(I : X) \hookrightarrow BS^p(I : X),$$

where the inclusions are strict in the set-theoretical sense.

2. Mixed Lebesgue spaces

Let us recall that the notion of a mixed Lebesgue space can be traced back to the paper of L. Hörmander [17], where he investigated the estimates for translation invariant operators (1960). The mixed Lebesgue spaces (or the Lebesgue spaces with vector exponents $L^{\vec{p}}$) are considered as a natural generalization of the classical Lebesgue space L^p via replacing the constant exponent p by a vector exponent $\vec{p} := (p_1, ..., p_n) \in (0, +\infty]^n$. A first detailed study of the mixed Lebesgue spaces is carried out by A. Benedek and R. Panzone in [18] (1961); see also [19–26] and references cited therein for more details about the subject.

Definition goes as follows. For any $\vec{p} := (p_1, ..., p_n) \in [1, +\infty]^n$, we denote by $\overrightarrow{q} := (q_1, ..., q_n)$ its conjugate exponent, i. e., $\frac{1}{\vec{p}} + \frac{1}{\overrightarrow{q}} = 1$; namely, for any $j \in \{1, ..., n\}$, $\frac{1}{p_j} + \frac{1}{q_j} = 1$. If $\vec{p} = (p_1, ..., p_n)$ and $\overrightarrow{q} = (q_1, ..., q_n)$ are two vectors in $[1, +\infty]^n$, then we write $\vec{p} \leqslant \overrightarrow{q}$ ($\vec{p} < \overrightarrow{q}$) if and only if $p_j \leqslant q_j$ ($p_j < q_j$) for any $j \in \{1, ..., n\}$.

Definition 2. Let $\emptyset \neq \Lambda_j \subseteq \mathbb{R}$ be a Lebesgue measurable set ($1 \leqslant j \leqslant n$), let $\Lambda = \prod_{j=1}^n \Lambda_j \subseteq \mathbb{R}^n$, and let $\vec{p} = (p_1, ..., p_n) \in [1, +\infty]^n$. The *mixed Lebesgue space* $L^{\vec{p}}(\Lambda : X)$ is defined to be the set of all Lebesgue measurable functions $F : \Lambda \to X$ such that

$$\|f\|_{L^{\vec{p}}(\Lambda:X)} := \left(\int_{\Lambda_n} \dots \left(\int_{\Lambda_2} \left(\int_{\Lambda_1} \|F(s_1, ..., s_n)\|_X^{p_1} \, ds_1 \right)^{p_2/p_1} ds_2 \right)^{p_3/p_2} \dots ds_n \right)^{1/p_n} < \infty,$$

with the usual modifications when $p_j = +\infty$ for some $j \in \{1, ..., n\}$.

In the case that $p_1 = ... = p_n = p$, with some $p \in [1, +\infty]$, the space $L^{\vec{p}}(\Lambda : X)$ coincides with the usual Lebesgue space $L^p(\Lambda : X)$.

Applying successively Minkowski's inequality, for F, $G \in L^{\vec{p}}(\Lambda : X)$, $\vec{p} \in [1, +\infty]^n$, we obtain the following Minkowski's inequality in $L^{\vec{p}}(\Lambda : X)$:

$$\|F + G\|_{L^{\vec{p}}(\Lambda:X)} \leqslant \|F\|_{L^{\vec{p}}(\Lambda:X)} + \|G\|_{L^{\vec{p}}(\Lambda:X)}. \tag{1}$$

Similarly, for $F \in L^{\vec{p}}(\Lambda : X)$ and $G \in L^{\overrightarrow{q}}(\Lambda : X)$, we have $FG \in L^1(\Lambda : X)$ and the successive applications of the usual Hölder inequality gives the following Hölder inequality in $L^{\vec{p}}(\Lambda : X)$:

$$\|FG\|_{L^1(\Lambda:X)} \leqslant \|F\|_{L^{\vec{p}}(\Lambda:X)} \ \|G\|_{L^{\overrightarrow{q}}(\Lambda:X)}, \tag{2}$$

for any $\vec{p} \in [1,+\infty]^n$ and $\overrightarrow{q} \in [1,+\infty]^n$ satisfying $\frac{1}{\vec{p}} + \frac{1}{\vec{q}} = 1$. As a consequence, we have that $(L^{\vec{p}}(\Lambda : X), \|\cdot\|_{L^{\vec{p}}(\Lambda:X)})$ is a Banach space for any $\vec{p} \in [1,+\infty]^n$.

Before proceeding to the next section, we would like to recall that the mixed Lebesgue spaces play an important role in the abstract harmonic analysis, especially in the theory of Wiener amalgam spaces and the theory of modulation spaces. For example, the mixed Lebesgue norm appears in definitions of the generalized modulation space $M^m_{p,q}(\mathbb{R}^d)$ introduced by H. Feichtinger and K. Gröchening in [27, Definition 2.3], the amalgam space $W(L^p; L^q_\omega)$ introduced by C. Heil in [28, Definition 11.3.1], the mixed Lebesgue space $L^{p,q}(v)$ introduced by H. Rauhut in [29, Section 6], and the general ultramodulation space $M^{\omega_\gamma}_{p,q}$ introduced by N. Teofanov in [30, Definition 4, p. 36]; see also the research monograph [31] by K. Gröchenig for more details about the subject.

3. Stepanov-like almost periodicity in mixed Lebesgue spaces

In this section, we will extend the concept of S^p-almost periodicity to the Lebesgue spaces $L^{\vec{p}}(\Lambda : X)$ with vector exponent $\vec{p} \in [1,+\infty)^n$. Unless stated otherwise, in the sequel we will always assume that $\Omega = [0,1]^n$.

First of all, we will investigate the notion of Stepanov $\vec{p}$-boundedness.

3.1. Stepanov $\vec{p}$-bounded functions depending on two parameters

We start this subsection by introducing the following notion:

Definition 3. Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ and $\vec{p} = (p_1, ..., p_n) \in [1,+\infty)^n$. A function $F : \Lambda \times X \to Y$ is said to be *$\vec{p}$-locally integrable on $\mathcal{B}$* if and only if, for every $B \in \mathcal{B}$ and for every sequence $(K_j)_{1\leqslant j\leqslant n}$ of compact subsets of $\mathbb{R}$ such that $K_1 \times K_2 \times \cdots \times K_n \subseteq \Lambda$, we have:

$$\|F\|_{L^{\vec{p}}(\Lambda:X),(K_j)_{1\leqslant j\leqslant n},B} :=$$
$$:= \sup_{x\in B} \left(\int_{K_n} ... \left(\int_{K_2} \left(\int_{K_1} \|F(s_1, ..., s_n; x)\|_X^{p_1} \, ds_1 \right)^{p_2/p_1} ds_2 \right)^{p_3/p_2} ... ds_n \right)^{1/p_n} < \infty.$$

The set of all $\vec{p}$-locally integrable functions on Λ is denoted by $L^{\vec{p},\mathcal{B}}_{loc}(\Lambda \times X : Y)$.

Let us recall [16] that the multi-dimensional Bochner transform $F^b : \Lambda \times X \to Y^\Omega$ of a function $F : \Lambda \times X \to Y$ is defined by

$$\left[F^b(\mathbf{t}; x)\right](\mathbf{u}) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

Now we are ready to introduce the following notion:

Definition 4. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda+[0,1]^n \subseteq \Lambda$ and let $\vec{p} = (p_1, ..., p_n) \in [1,+\infty)^n$. Let a function $F : \Lambda \times X \to Y$ be $\vec{p}$-locally integrable on $\mathcal{B}$. Then we say that $F(\cdot;\cdot)$ is *Stepanov $\vec{p}$-bounded* on $\mathcal{B}$ if and only if for every $B \in \mathcal{B}$ there exists a finite real constant $M > 0$ such that

$$\|F\|_{S^{\vec{p},B}} := \sup_{\mathbf{t}\in\Lambda;x\in B}\left(\int_0^1\left(\int_0^1\left(\int_0^1\left\|F\left(t_1+s_1,...,t_n+s_n;x\right)\right\|^{p_1}ds_1\right)^{p_2/p_1}ds_2\right)^{p_3/p_2}...\,ds_n\right)^{1/p_n} < M$$

for any $\mathbf{t} = (t_1, ..., t_n) \in \Lambda$ and $x \in B$. The collection of these functions will be denoted by $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$.

By applying Minkowski's inequality (1), it is easy to show that $L^{\vec{p},\mathcal{B}}_{loc}(\Lambda \times X : Y)$ and $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ are vector spaces. Let $F(\cdot;\cdot)$ be Stepanov $\vec{p}$-bounded on $\mathcal{B}$, and let $B \in \mathcal{B}$ be fixed. Then it is easy to see that $\|\cdot\|_{S^{\vec{p},B}}$ is a norm on $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$; furthermore, we have the following expected result:

Proposition 1. *Let* $\vec{p} \in [1, +\infty)^n$. *Then* $(BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y), \|\cdot\|_{S^{\vec{p},B}})$ *is a Banach space.*

Proof. Let $(F_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$. Then we have:

$$\forall\varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall i, j \geqslant j_0 \Rightarrow \|F_i - F_j\|_{S^{\vec{p},B}} \leqslant \varepsilon.$$

This shows that $(F_j(\mathbf{t}; x))_{j\in\mathbb{N}}$ is a Cauchy sequence in the Banach space

$$L^{\vec{p}}(\prod_{k=1}^{n}[t_k, t_k + 1] : Y)$$

uniformly with respect to $\mathbf{t} = (t_1, ..., t_n) \in \Lambda$ and $x \in B$, so there exists a function $F : \Lambda \times X \to Y$ such that the value

$$\sup_{(t_1,...,t_n)\in\Lambda;x\in B}\left(\int_{t_n}^{t_n+1}...\left(\int_{t_1}^{t_1+1}\|F_j\left(s_1,...,s_n;x\right)-F\left(s_1,...,s_n;x\right)\|_X^{p_1}ds_1\right)^{p_2/p_1}...ds_n\right)^{1/p_n}$$

tends to zero. Minkowski's inequality allows us to conclude that:

$$\|F\|_{S^{\vec{p},B}} \leqslant \|F_j - F\|_{S^{\vec{p},B}} + \|F_j\|_{S^{\vec{p},B}} < \infty, \quad j \in \mathbb{N},$$

which shows that $F \in BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ since $(F_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the space $BS^{\vec{p}}(\Lambda \times X : Y)$ and therefore bounded. □

Under certain very reasonable assumptions, we have that $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ is translation invariant in both arguments. Further on, if we assume that $\vec{1} \leqslant \vec{q} \leqslant \vec{p}$, then we can use the Hölder inequality (2) in order to see that there exist two finite real constants $c_1 > 0$ and $c_2 > 0$ such that the following estimates hold true

$$\|F\|_{S^{\vec{q},B}} \leqslant c_1\|F\|_{S^{\vec{p},B}} \leqslant c_2\|F\|_{S^{\vec{1},B}},$$

whenever the above expressions make a sense; here, of course, $\vec{1} := (1, 1, ..., 1)$.

3.2. Stepanov-like almost periodic functions in mixed Lebesgue spaces

Suppose that R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$. Following our recent considerations from [16], we introduce the following spaces of mixed Lebesgue — Stepanov-like almost periodic functions:

Definition 5. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the following condition holds:

$$\text{If } \ \mathbf{t} \in \Lambda, \ \mathbf{b} \in \mathrm{R} \ \text{ and } \ l \in \mathbb{N}, \ \text{ then we have } \ \mathbf{t} + \mathbf{b}(l) \in \Lambda.$$

Let the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ be well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is *Stepanov* $(\vec{p}, \mathrm{R}, \mathcal{B})$*-multi-almost periodic* if and only if the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ such that

$$\lim_{l \to +\infty} \Bigl\| F\bigl(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \cdots, b_{k_l}^n); x\bigr) - \Bigl[F^*(\mathbf{t}; x)\Bigr](\mathbf{u}) \Bigr\|_{L^{\vec{p}}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $APS^{\vec{p}}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$ we denote the collection consisting of all Stepanov $(\vec{p}, \mathrm{R}, \mathcal{B})$-multi-almost periodic functions $F : \Lambda \times X \to Y$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is *Stepanov* $(\vec{p}, \mathrm{R})$*-multi-almost periodic* and shorten $APS^{\vec{p}}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$ to $APS^{\vec{p}}_{\mathrm{R}}(\Lambda : Y)$.

Definition 6. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the following condition holds:

$$\text{If } \ \mathbf{t} \in \Lambda, \ (\mathbf{b}; \mathbf{x}) \in \mathrm{R_X} \ \text{ and } \ l \in \mathbb{N}, \ \text{ then we have } \ \mathbf{t} + \mathbf{b}(l) \in \Lambda.$$

Let the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ be well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is *Stepanov* $(\vec{p}, \mathrm{R_X}, \mathcal{B})$*-multi-almost periodic* if and only if the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is $(\mathrm{R_X}, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdots, b_k^n); x_k)) \in \mathrm{R_X}$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ such that

$$\lim_{l \to +\infty} \Bigl\| F\bigl(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \cdots, b_{k_l}^n); x + x_{k_l}\bigr) - \Bigl[F^*(\mathbf{t}; x)\Bigr](\mathbf{u}) \Bigr\|_{L^{\vec{p}}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $APS^{\vec{p}}_{(\mathrm{R_X},\mathcal{B})}(\Lambda \times X : Y)$ we denote the space consisting of all Stepanov $(\vec{p}, \mathrm{R_X}, \mathcal{B})$-multi-almost periodic functions.

Definition 7. Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a continuous function and $\Lambda + \Lambda' \subseteq \Lambda$. Then we say that:

(i) $F(\cdot;\cdot)$ is *Stepanov* $(\vec{p}, \mathcal{B}, \Lambda')$*-almost periodic* (*Stepanov* $(\vec{p}, \mathcal{B})$*-almost periodic*, if $\Lambda = \Lambda'$) if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\bigl\| F(\mathbf{t} + \tau + \cdot; x) - F(\mathbf{t} + \cdot; x) \bigr\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \epsilon, \quad \mathbf{t} \in \Lambda, \ x \in B.$$

(ii) $F(\cdot;\cdot)$ is *Stepanov* $(\vec{p},\mathcal{B},\Lambda')$*-uniformly recurrent* ($(\vec{p},\mathcal{B})$*-uniformly recurrent*, if $\Lambda=\Lambda'$) if and only if for every $B\in\mathcal{B}$ there exists a sequence (τ_n) in Λ' such that $\lim_{n\to+\infty}|\tau_n|=+\infty$ and

$$\lim_{n\to+\infty}\sup_{\mathbf{t}\in\Lambda;x\in B}\big\|F(\mathbf{t}+\tau_n+\cdot;x)-F(\mathbf{t}+\cdot;x)\big\|_{L^{\vec{p}}(\Omega:Y)}=0.$$

If $X\in\mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is *Stepanov* $(\vec{p},\Lambda')$*-almost periodic* ($(\vec{p},\Lambda')$*-uniformly recurrent*) [*Stepanov almost periodic* (*uniformly recurrent*), if $\Lambda=\Lambda'$].

The use of space $L^{\vec{p}}(\Omega:X)$ in our approach is new but very similar to the use of space $L^{p(\mathbf{u})}(\Omega:Y)$ in [16]. Keeping in mind the proofs of our structural results from [16], it becomes clear from the introduced notion and the fact that the Minkowski inequality and the Hölder inequality hold in mixed Lebesgue spaces, that many structural results from [16] remain true in our framework without any essential changes of the notion and notation. For example, the statements of [16, Proposition 2.6, Propositions 2.10–2.12, Theorems 2.13–2.15, Propositions 2.17–2.19, Proposition 2.25] can be straightforwardly formulated for the Stepanov classes of functions introduced in Definitions 5–7; similarly, the supremum formula for Stepanov-like almost periodic functions in mixed Lebesgue spaces, the relative compactness of range $\{F^b(\mathbf{t};x):\mathbf{t}\in\Lambda;\ x\in B\}$, for a given set $B\in\mathcal{B}$, and some results about the composition of Stepanov-like almost periodic functions in mixed Lebesgue spaces, can be achieved similarly as in [16]. In particular, it is worth noting that any Stepanov $(\vec{p},\mathcal{B})$-almost periodic function defined on $\mathbb{R}^n$ must be Stepanov $\vec{p}$-bounded, provided that $\mathcal{B}$ is a collection of compact subsets of X, as well as that, in this case, the space $APS^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n:Y)$, consisting of all Stepanov $(\vec{p},\mathcal{B})$-almost periodic functions $F:\mathbb{R}^n\times X\to Y$, is densely and continuously embedded in the space $BS^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n:Y)$.

The result about the convolution invariance of Stepanov-like almost periodic functions in mixed Lebesgue spaces, which can be simply obtained by reformulating [16, Proposition 2.10], can be applied to the Gaussian semigroup in $\mathbb{R}^n$; see [16, Section 6] for more details about applications of multi-dimensional Stepanov functions, given especially in the case that $\vec{p}=\vec{1}$.

For the sake of completeness, we will present all details of the proof of the following analogue of [16, Proposition 2.22] (proposed by our friend and colleague Kamal Khalil from Marrakech, Morocco):

Proposition 2. *Let* $\Lambda+\Lambda\subseteq\Lambda$, $\Lambda+\Omega\subseteq\Lambda$, $\mathcal{B}$ *is any family of compact subsets of* X *and* $F:\Lambda\times X\to Y$ *satisfy the following conditions:*

(i) *For each* $x\in X$*, the function* $F(\cdot;x)$ *is Stepanov* $(\vec{p},\Lambda)$*-almost periodic.*

(ii) *For each* $\varepsilon>0$ *and* $B\in\mathcal{B}$ *there exists* $\delta_{B,\varepsilon}>0$ *such that for all* $x_1,x_2\in B$ *one has*

$$\|x_1-x_2\|\leqslant\delta_{B,\varepsilon}\Longrightarrow\Big\|F(\mathbf{t}+\cdot;x_1)-F(\mathbf{t}+\cdot;x_2)\Big\|_{L^{\vec{p}}(\Omega:Y)}\leqslant\varepsilon\quad\textit{for all}\ \ \mathbf{t}\in\Lambda.$$

Then $F(\cdot;\cdot)$ *is Stepanov* $(\vec{p},\mathcal{B})$*-almost periodic.*

Proof. Let $\varepsilon>0$ and $B\subseteq X$ be a compact set. Then there exists a finite subset $\{x_1,...,x_n\}\subseteq B$ ($n\in\mathbb{N}$) such that $B\subseteq\bigcup_{i=1}^n B(x_i,\delta_{B,\varepsilon})$. Hence, for every $x\in B$, there exists $i\in\mathbb{N}_n\equiv\{1,2,...,n\}$ satisfying $\|x-x_i\|\leqslant\delta_{B,\varepsilon}$. Let $\tau\in\Lambda$. Then we have

$$\begin{aligned}\|F(\mathbf{t}+\cdot+\tau;x)-F(\mathbf{t}+\cdot;x)\|_{L^{\vec{p}}(\Omega:Y)}&\leqslant\|F(\mathbf{t}+\cdot+\tau;x)-F(\mathbf{t}+\cdot+\tau;x_i)\|_{L^{\vec{p}}(\Omega:Y)}+\\+\|F(\mathbf{t}+\cdot+\tau;x_i)-F(\mathbf{t}+\cdot;x_i)\|_{L^{\vec{p}}(\Omega:Y)}&+\|F(\mathbf{t}+\cdot;x_i)-F(\mathbf{t}+\cdot;x)\|_{L^{\vec{p}}(\Omega:Y)},\ \mathbf{t}\in\Lambda.\end{aligned}\tag{3}$$

Using (i), we have that for each $i = 1, \ldots, n$ there exists $l_{B,\varepsilon} > 0$ such that for all $\mathbf{t}_0 \in \Lambda$ there exists $\tau \in B(\mathbf{t}_0, l_{B,\varepsilon})$ satisfying

$$\|F(\mathbf{t} + \cdot + \tau; x_i) - F(\mathbf{t} + \cdot; x_i)\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{4}$$

Since $\|x - x_i\| \leqslant \delta_{K,\delta}$, by (ii) we get

$$\|F(\mathbf{t} + \cdot + \tau; x) - F(\mathbf{t} + \cdot + \tau; x_i)\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda, \tag{5}$$

and

$$\|F(\mathbf{t} + \cdot; x) - F(\mathbf{t} + \cdot; x_i)\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{6}$$

Inserting (4), (5) and (6) in (3), we get

$$\sup_{x \in B} \|F(\mathbf{t} + \cdot + \tau; x) - F(\mathbf{t} + \cdot; x)\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \varepsilon \quad \text{for all } \mathbf{t} \in \Lambda.$$

Hence, $F(\cdot; \cdot)$ is Stepanov $(\vec{p}, \mathcal{B})$-almost periodic. □

The following statement can be formulated for all other classes of functions introduced in Definitions 5–7:

Proposition 3. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is well defined and continuous. If $\vec{1} \leqslant \vec{q} \leqslant \vec{p}$, then*

$$APS^{\vec{q}}_{\mathcal{B}}(\mathbb{R}^n : Y) \subseteq APS^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n : Y) \subseteq APS^{\vec{1}}_{\mathcal{B}}(\mathbb{R}^n : Y).$$

Using Proposition 3 and [16, Theorem 2.21] with $p(\mathbf{u}) \equiv 1$, we immediately get the following (see [2] for the notion of Bohr $\mathcal{B}$-almost periodicity):

Proposition 4. *Suppose that $\mathcal{B}$ is any family of compact subsets of X. If $F : \mathbb{R}^n \times X \to Y$ is uniformly continuous and Stepanov $(\vec{p}, \mathcal{B})$-almost periodic, then $F(\cdot; \cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

Furthermore, we can use Proposition 3 and our analysis from [16, Example 2.9] in order to conclude that for each $\vec{p} \in [1, \infty)^n$ we have the following (just take $p(\mathbf{u}) \equiv \max\{p_i : 1 \leqslant i \leqslant n\}$ in the above mentioned-example; see [16] for the notions of Bohr Λ'-almost periodicity and Bohr Λ'-uniform recurrence):

Example 1. Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is a Bohr Λ'-almost periodic function (Λ'-uniformly recurrent function). Define $\operatorname{sign}(0) := 0$ and $H : \mathbb{R}^n \to \mathbb{R}$ by $H(\mathbf{t}) := \operatorname{sign}(F(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$. Then the function $H(\cdot)$ is Stepanov $(\vec{p}, \Lambda')$-almost periodic (Stepanov $(\vec{p}, \Lambda')$-uniformly recurrent), provided that

$$(\exists L \geqslant 1)\ (\forall \epsilon > 0)\ \big(\forall y \in \mathbb{R}^n\big)\ m\big(\{x \in y + \Omega : |F(x)| \leqslant \epsilon\}\big) \leqslant L\epsilon.$$

In particular, the last estimate holds for any multivariate trigonometric polynomial.

In this paper, we will not consider the composition principles for Stepanov-like almost periodic functions in mixed Lebesgue spaces, as well as the invariance of Stepanov-like almost periodicity in mixed Lebesgue spaces under the actions of convolution products. Asymptotically Stepanov-like almost periodic functions in mixed Lebesgue spaces will not be considered, as well (see [2, Sections 3–5] for more details). The idea of using space $L^{\vec{p}}(\Omega : X)$ can be also interesting for the study of corresponding classes of multi-dimensional Weyl almost periodic functions in mixed Lebesgue spaces [13]; this theme will be analyzed somewhere else. Let us also recall that, for any given number $p > 1$, H. Bohr and E. Fölner have constructed a Stepanov 1-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ which is not Stepanov p-almost periodic (see [32, Example, p. 70]). The analysis of such non-trivial examples for multi-dimensional Stepanov almost periodic functions analyzed in [2] and this paper will be also carried out somewhere else.

Finally, we will present the following illustrative application to close the whole paper (see also [13]; the final conclusion of Example 2 can be deduced by assuming that $\vec{p} = \vec{1}$ from the very beginning):

Example 2. Suppose that $Y := L^r(\mathbb{R}^n)$ for some $r \in [1, \infty)$ and $A(t) := \Delta + a(t)I$, $t \geqslant 0$, where Δ is the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$, I is the identity operator on $L^r(\mathbb{R}^n)$ and $a \in L^\infty([0, \infty))$. Then we know that the evolution system $(U(t, s))_{t \geqslant s \geqslant 0} \subseteq L(Y)$ generated by the family $(A(t))_{t \geqslant 0}$ exists and is given by $U(t, t) := I$ for all $t \geqslant 0$ and

$$[U(t, s)F](\mathbf{u}) := \int_{\mathbb{R}^n} K(t, s, \mathbf{u}, \mathbf{v})F(\mathbf{v})\, d\mathbf{v}, \quad F \in L^r(\mathbb{R}^n), \quad t > s \geqslant 0,$$

where $K(t, s, \mathbf{u}, \mathbf{v})$ is given through

$$K(t, s, \mathbf{u}, \mathbf{v}) := (4\pi(t - s))^{-n/2} e^{\int_s^t a(\tau)\, d\tau} \exp\left(-\frac{|\mathbf{u} - \mathbf{v}|^2}{4(t - s)}\right), \quad t > s,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n.$$

It is clear that, for every $\tau \in \mathbb{R}^n$, we have:

$$K(t, s, \mathbf{u} + \tau, \mathbf{v} + \tau) = K(t, s, \mathbf{u}, \mathbf{v}), \quad t > s \geqslant 0,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n.$$

Further on, under certain assumptions, a unique mild solution of the abstract Cauchy problem $(\partial/\partial t)u(t, x) = A(t)u(t, x)$, $t > 0$; $u(0, x) = F(x)$ is given by $u(t, x) := [U(t, 0)F](x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Suppose now that $F : \mathbb{R}^n \to \mathbb{C}$ is Stepanov $(\vec{p}, \mathbb{R}^n)$-almost periodic. Let $1/\vec{p} + 1/\vec{q} = 1$, and let $t > 0$ be fixed. Then there exists a finite real constant $c_t > 0$ such that, for every $\mathbf{t},\ \tau \in \mathbb{R}^n$ and $\mathbf{u} \in \Omega$, we have:

$$|u(t, \mathbf{t} + \mathbf{u} + \tau) - u(t, \mathbf{t} + \mathbf{u})| = \left| \int_{\mathbb{R}^n} \big[K(t, 0, \mathbf{t} + \mathbf{u} + \tau, \mathbf{v}) - K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v})\big] F(\mathbf{v})\, d\mathbf{v} \right| =$$

$$= \left| \int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u} + \tau, \mathbf{v} + \tau) F(\mathbf{v} + \tau)\, d\mathbf{v} - \int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v}) F(\mathbf{v})\, d\mathbf{v} \right| =$$

$$= \left| \int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v}) \big[F(\mathbf{v} + \tau)\, d\mathbf{v} - F(\mathbf{v})\big]\, d\mathbf{v} \right| \leqslant$$

$$\leqslant c_t \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{t} + \mathbf{u} - \mathbf{v}|^2}{4t}} |F(\mathbf{v} + \tau) - F(\mathbf{v})|\, d\mathbf{v} =$$

$$= c_t \sum_{k \in \mathbb{Z}^n} \int_{[0,1]^n} e^{-\frac{|\mathbf{t} + \mathbf{u} - \mathbf{v} - k|^2}{4t}} |F(k + \mathbf{v} + \tau) - F(k + \mathbf{v})|\, d\mathbf{v} \leqslant$$

$$\leqslant c_t \sum_{k \in \mathbb{Z}^n} \left\| e^{-\frac{|\mathbf{t} + \mathbf{u} - \cdot - k|^2}{4t}} \right\|_{L^{\vec{q}}([0,1]^n)} \left\| F(k + \cdot + \tau) - F(k + \cdot) \right\|_{L^{\vec{p}}([0,1]^n)} \leqslant$$

$$\leqslant c_t \sum_{k \in \mathbb{Z}^n} \left\| e^{-\frac{|\mathbf{t} + \mathbf{u} - \cdot - k|^2}{4t}} \right\|_{L^\infty([0,1]^n)} \left\| F(k + \cdot + \tau) - F(k + \cdot) \right\|_{L^{\vec{p}}([0,1]^n)}.$$

If $\tau \in \mathbb{R}^n$ satisfies $\|F(\mathbf{t}+\cdot+\tau)-F(\mathbf{t}+\cdot)\|_{L^{\vec{p}}([0,1]^n)} < \epsilon$ for all $\mathbf{t} \in \mathbb{R}^n$, then the above implies

$$|u(t,\mathbf{t}+\mathbf{u}+\tau)-u(t,\mathbf{t}+\mathbf{u})| \leqslant c_t\epsilon \sum_{k\in\mathbb{Z}^n} \Big\| e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}} \Big\|_{L^\infty([0,1]^n)}, \quad \mathbf{t}\in\mathbb{R}^n,\ \mathbf{u}\in\Omega.$$

A very simple computation involving the Cauchy — Schwartz inequality shows that

$$\Big\| e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}} \Big\|_{L^\infty([0,1]^n)} \leqslant e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}}, \quad \mathbf{t}\in\mathbb{R}^n,\ k\in\mathbb{Z}^n,\ \mathbf{u}\in\Omega,$$

so that

$$|u(t,\mathbf{t}+\mathbf{u}+\tau)-u(t,\mathbf{t}+\mathbf{u})| \leqslant c_t\epsilon \sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}}, \quad \mathbf{t}\in\mathbb{R}^n,\ \mathbf{u}\in\Omega. \tag{7}$$

Since the function defined by the above series is continuous in the variable $\mathbf{t}\in\mathbb{R}^n$, there exists a finite real number $M_t \geqslant 1$ such that

$$\sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}} \leqslant M_t, \quad |\mathbf{t}|\leqslant 1;$$

furthermore, if $|\mathbf{t}|>1$, then we have

$$\sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}} \leqslant \sum_{k\in\mathbb{Z}^n} e^{-\frac{|k|^2-2|k||\mathbf{t}|+|\mathbf{t}|^2-4\sqrt{n}|k|-4\sqrt{n}|\mathbf{t}|}{4t}} \leqslant e^{-\frac{|\mathbf{t}|^2-4\sqrt{n}|\mathbf{t}|}{4t}} \sum_{k\in\mathbb{Z}^n} e^{-\frac{|k|^2-2|k|-4\sqrt{n}||k|}{4t}},$$

which simply implies along with the estimate (7) that the function $x\mapsto u(t,x)$, $x\in\mathbb{R}^n$ is Bohr almost periodic in the usual sense since it is continuous.

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ПОЧТИ ПЕРИОДИЧНОСТЬ ТИПА СТЕПАНОВА В СМЕШАННЫХ ПРОСТРАНСТВАХ ЛЕБЕГА

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Главная цель работы — пересмотреть недавно изученный класс многомерных степановских почти периодических функций. Мы вводим в рассмотрение и исследуем новые классы почти периодических функций типа Степанова в смешанных пространствах Лебега. Мы также даём новое приложение многомерных степановских почти периодических функций к абстрактным дифференциальным уравнениям первого порядка в случае, когда все компоненты $\vec{p} \in [1, \infty)^n$ равны.

Ключевые слова: *почти периодическая функция типа Степанова в смешанном пространстве Лебега, многомерная почти периодическая функция Степанова, абстрактное неавтономное дифференциальное уравнение первого порядка.*

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