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DYNAMICS OF A FAMILY OF MAPS DEFINED BY QUADRATIC POLYNOMIALS

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We consider maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, whose coordinates are homogeneous polynomials in $\mathbb{R}[x,y]$ of degree 2. These maps send lines passing through the origin into lines passing through the origin. Our goal is to study how these lines are moved under the action of F. We show that there is a real analytic variety $\mathcal{F}^2$, where two sets can be clearly distinguished. One set $\mathcal{U} \subseteq \mathcal{F}^2$ is made up of transformations that have "hidden hyperbolic" dynamics, and its complement $\mathcal{F}^2 \setminus \mathcal{U}$ contains maps that show a chaotic behavior.

Keywords: *polynomial map, circle map, chaotic dynamics.*

1. Introduction and Main Result

Let us consider maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P,Q)$, where $P,Q \in \mathbb{R}[x,y]$ are homogeneous polynomials of degree $k \geqslant 1$, which have no real lines in common, that is, P and Q have no linear factors with real coefficients in common. These maps send lines passing through the origin into lines passing through the origin. In a broad view, our goal is to study how these lines are moved under the action of F.

To approach formally our problem, we can identify all lines in $\mathbb{R}^2$ passing through the origin with the projective space $\mathbb{RP}^1$, which is a circle. Assuming, for the moment intuitively, that the considered maps F induce *circle maps* (i. e., maps from the circle to itself), our problem is equivalent to study the dynamics of such circle maps.

Considering circle maps that turn out to be homeomorphisms, H. Poincaré from his early studies explored when a circle homeomorphism is equivalent to a rotation, that is, under which conditions a circle homeomorphism and a rotation are conjugated. In the case when the circle map is an orientation-preserving homeomorphism, this problem can be studied associating with this circle map its *rotation number* (see [1, Definition 11.1.2]). Intuitively, this number determines the average displacement of any point in the circle by the circle map. We can say that this number determines the rotation that looks like the circle map, on average. It is worthwhile to point that the rotation number is invariant under conjugation by orientation-preserving homeomorphisms (see [1, Proposition 11.1.3]).

Poincaré accomplished a comprehensive study of the dynamics of orientation-preserving homeomorphisms in terms of the rotation number ([1, Chapter 11, §2]). In fact, when the rotation number is irrational, he obtained an answer to his initial problem. Poincaré concluded that when the homeomorphism with irrational rotation number is a transitive map, then it is *topologically equivalent* to the respective rotation, that is, the conjugation is realized by a homeomorphism.

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When a circle map is not a homeomorphism, in general, there is not an invariant that, as the rotation number, enables us to describe the behavior of the map. Then the dynamics in these cases require more specific studies. In our case, we focus on the study of a family of maps in $\mathbb{R}^2$ that induce noninvertible circle maps, that is, circle maps that are not homeomorphisms.

More specifically, we consider maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, whose coordinates are homogeneous polynomials in $\mathbb{R}[x,y]$ of degree $k=2$. We focus on a subfamily of these maps, $\mathcal{F}^2$, which turns out to be a real analytic variety. We describe an open subset $\mathcal{U} \subseteq \mathcal{F}^2$ of this variety, where each map $F \in \mathcal{U}$ shows "hidden hyperbolic" dynamics that moves the lines passing through the origin in a simple way. We also show that in the complement, $\mathcal{F}^2 \setminus \mathcal{U}$, there exist maps moving with a chaotic behavior these lines. In particular, the given descriptions enable us to understand how the points in the basin of attraction of the origin are moved around this fixed point.

1.1. The Statement of the Principal Result

In this subsection we state our main result in a simple way, avoiding to introduce several notions. Later, in Section 2, we will provide a suitable context which enables us, among other things, to prove our main result formally.

Let $\mathcal{F}^2$ be the family of all maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F=(P,Q)$, where $P,Q \in \mathbb{R}[x,y]$ are homogeneous polynomials of degree $k=2$, which have no real lines in common, and such that there exist exactly three pairwise distinct real lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ passing through the origin, each of them invariant under F.

Our main result is Theorem 1, which we state below using the following terms: let $\mathcal{L}$, $\mathcal{R}$ be two different lines in $\mathbb{R}^2$ passing through the origin. Let us denote with $\mathrm{C}(\mathcal{L},\mathcal{R})$ the set of all real lines in $\mathbb{R}^2$ passing through the origin, that are covered when we move, in the counterclockwise (positive) direction, from the position of $\mathcal{L}$ to the position of $\mathcal{R}$. Let us stress that $\mathrm{C}(\mathcal{L},\mathcal{R})$ contains both lines $\mathcal{L}$, $\mathcal{R}$, and it is a closed subset of $\mathbb{R}^2$.

Theorem 1. *The family of maps $\mathcal{F}^2$ is a real analytic variety. This family has an open subset $\mathcal{U}$ where each mapping $F \in \mathcal{U}$ satisfies the following properties: given $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ the three invariant lines under F.*

(i) *There exists a real line $\mathcal{R}$ passing through the origin, different from the invariant line $\mathcal{L}_3$, such that $\mathcal{R}$ is mapped to $\mathcal{L}_3$ by F.*

(ii) *The sets $\mathrm{C}(\mathcal{L}_3,\mathcal{R})$ and $\mathrm{C}(\mathcal{R},\mathcal{L}_3)$ are invariant under F.*

(iii) *The other two invariant lines $\mathcal{L}_1$, $\mathcal{L}_2$ are contained in the two invariant sets $\mathrm{C}(\mathcal{L}_3,\mathcal{R})$ and $\mathrm{C}(\mathcal{R},\mathcal{L}_3)$. Moreover, $\mathcal{L}_1 \subseteq \mathrm{C}(\mathcal{L}_3,\mathcal{R})$, and $\mathcal{L}_2 \subseteq \mathrm{C}(\mathcal{R},\mathcal{L}_3)$.*

(iv) *Let $(x,y) \in \mathbb{R}^2$ be a point which is not in the lines $\mathcal{L}_3$, $\mathcal{R}$. If (x,y) is in $\mathrm{C}(\mathcal{L}_3,\mathcal{R})$, then its orbit under F approaches the invariant line $\mathcal{L}_1$; if (x,y) is in $\mathrm{C}(\mathcal{R},\mathcal{L}_3)$, then its orbit under F approaches the invariant line $\mathcal{L}_2$.*

On the other hand, in the complement $\mathcal{F}^2 \setminus \mathcal{U}$ of the open subset $\mathcal{U}$, there exist maps where the dynamics of the lines is chaotic (in the sense of Devaney).

Remark 1. For each $F \in \mathcal{F}^2$, the origin of $\mathbb{R}^2$ is a fixed point. Since the derivative of F at the origin is identically zero, then the origin is an attractor. In addition, we assume that the mapping F belongs to the open subset $\mathcal{U}$ indicated in Theorem 1. Then, the orbit of a point in the basin of attraction of the origin converges to the origin, approaching one of the three invariant lines of the map F.

Considering the open subset $\mathcal{U}$ of the family $\mathcal{F}^2$ mentioned in Theorem 1, the mappings $F \in \mathcal{U}$ move the lines in $\mathbb{R}^2$ passing through the origin in a simple way.

On the other hand, in the complement $\mathcal{F}^2 \setminus \mathcal{U}$ of the open subset $\mathcal{U}$, the maps do not always have simple dynamics, as we shall see with an example of a map moving with a chaotic behavior (in the sense of Devaney) these lines.

In Section 2, we approach these problems and results in a precise way. Namely, we introduce an appropriate variety and several notions, which shall provide an appropriate context to analyse the dynamics of the lines under a mapping in $\mathcal{F}^2$. In particular, in Subsection 2.3, we define the open subset $\mathcal{U}$ of the family $\mathcal{F}^2$, and we reformulate Theorem 1. In that subsection, the structure of the rest of the work is described in general terms.

2. Preliminary Notions and Reformulation of Theorem 1

In subsection 2.1 we introduce a real analytic variety $\mathbb{M}$, *the Möbius band*, which is formed by the punctured real plane $\mathbb{R}^2 \setminus \{0\}$, placing instead of the origin, the space of directions $\mathbb{RP}^1$. In Subsection 2.2, we define *the blow-up of a map*, which enables us to study the dynamics of the mapping F, in particular, how the map F moves the lines in $\mathbb{R}^2$ passing through the origin. Finally, in Subsection 2.3, we give a reformulation of Theorem 1 using the preliminary notions. Furthermore, we outline its proof and make some comments on the general structure of the rest of the work.

2.1. Möbius Band

We consider the map from $\mathbb{R}^2 \setminus \{0\}$ to the projective space $\mathbb{RP}^1$, that sends a point $\mathbb{R}^2 \setminus \{0\}$ to the *slope* of the line generated by this point. Let M be the graph of this mapping contained in $\mathbb{R}^2 \times \mathbb{RP}^1$; we denote by $\mathbb{M}$ its closure, that is, $\mathbb{M} = M \cup \mathbb{D}$, where $\mathbb{D} := \{0\} \times \mathbb{RP}^1$. The set $\mathbb{M}$ is a real analytic variety with the charts (x, u), (v, y), and the transition maps $y = ux$ and $v = u^{-1}$, $u \neq 0$. In these charts, the circle $\mathbb{D}$ corresponds to $\{x = 0\}$ and $\{y = 0\}$, respectively. The variety $\mathbb{M}$ is called *the Möbius band*; the circle $\mathbb{D}$ is called *the exceptional divisor*, and it will be identified with $\mathbb{RP}^1$. As a consequence, the charts on the divisor are u, v, with $v = u^{-1}$.

We consider the projection $\mathbb{R}^2 \times \mathbb{RP}^1 \twoheadrightarrow \mathbb{R}^2$ on the first component, and denote by π its restriction to $\mathbb{M}$. Then, the projection $\pi\colon \mathbb{M} \twoheadrightarrow \mathbb{R}^2$ is a real analytic map, and its restriction to $\mathbb{M} \setminus \mathbb{D}$ is a real analytic diffeomorphism onto $\mathbb{R}^2 \setminus \{0\}$. In the charts (x, u), (v, y), the projection π is given by the mappings

$$\pi\colon (x, u) \longmapsto (x, y), \quad y = ux\,, \qquad \pi\colon (v, y) \longmapsto (x, y), \quad x = vy\,.$$

Given a neighborhood $U \subseteq \mathbb{R}^2$ of the origin, its preimage $\tilde{U} := \pi^{-1}(U)$ is called *the blow-up of* U.

Definition 1. Given two different points $\alpha, \beta \in \mathbb{D}$, we denote by $[\alpha, \beta]$ the segment of the divisor $\mathbb{D}$, obtained by going from α to β with the positive direction. In a similar way, we define the segments $(\alpha, \beta]$, $[\alpha, \beta)$, (α, β).

Remark 2. Let $\mathcal{L}$, $\mathcal{R}$ be two distinct lines in $\mathbb{R}^2$ passing through the origin; we denote by α, $\tilde{\alpha} \in \mathbb{D}$ their respective slopes. We consider C the set of all points in $\mathbb{M} \subseteq \mathbb{R}^2 \times \mathbb{RP}^1$ having the second coordinate in the segment $[\alpha, \tilde{\alpha}]$. Then the image of C under the projection $\pi\colon \mathbb{M} \to \mathbb{R}^2$ is equal to $\mathrm{C}(\mathcal{L}, \mathcal{R})$. On the other hand, the set of all points of $\mathbb{M}$ having the second coordinate in the segment $[\tilde{\alpha}, \alpha]$, has the image $\mathrm{C}(\mathcal{R}, \mathcal{L})$ under π.

2.2. Blow-up of a Map in $\mathbb{R}^2$

Let $U \subseteq \mathbb{R}^2$ be an open neighborhood of the origin in $\mathbb{R}^2$. Let $F\colon U \to \mathbb{R}^2$ be a smooth map (i. e. $\mathcal{C}^\infty$ map) such that $F(0) = 0$. We consider a smooth mapping $\tilde{F}\colon \tilde{U} \to \mathbb{M}$ from $\tilde{U}$, the blow-up of U, to the Möbius band $\mathbb{M}$. The map $\tilde{F}$ will be called *blow-up of* F if it is the pullback of F by the projection $\pi\colon \mathbb{M} \to \mathbb{R}^2$, that is, $\pi \circ \tilde{F} = F \circ \pi$.

The composition $\pi^{-1} \circ F \circ \pi$ is well defined at the points $p \in \tilde{U}$ satisfying $F \circ \pi(p) \neq 0$. Therefore, the existence of the blow-up of F lies in the fact that the composition $\pi^{-1} \circ F \circ \pi$ has a smooth extension to the whole open subset $\tilde{U}$, and in particular, to the exceptional divisor $\mathbb{D}$. Let us stress that the uniqueness of the blow-up occurs if the set of points $p \in \tilde{U}$ such that $F \circ \pi(p) \neq 0$ is a dense subset of $\tilde{U}$.

We focus on mappings $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P, Q)$, where $P, Q \in \mathbb{R}[x, y]$ are homogeneous polynomials of degree $k \geqslant 1$ which have no real lines in common. In this case, the composition $\pi^{-1} \circ F \circ \pi$ is well defined outside the divisor $\mathbb{D}$. In what follows, we verify that this composition is analytically extended to the divisor $\mathbb{D}$. For that purpose, we analyse the composition from the charts (x, u), (v, y) of the Möbius band $\mathbb{M}$.

The composition $\pi^{-1} \circ F \circ \pi$, from points in the chart (x, u) to points in the same chart, is represented by

$$(x, u) \mapsto \left(x^k P(1, u), \frac{Q(1, u)}{P(1, u)}\right) =: \left(x^k P(1, u), f(u)\right).$$

That composition, from points in the chart (x, u) to points in the chart (v, y), is represented by $(x, u) \mapsto \left((f(u))^{-1}, x^k Q(1, u)\right)$. Therefore, from the chart (x, u), the composition $\pi^{-1} \circ F \circ \pi$ is analytically extended to the divisor $\mathbb{D}$.

On the other hand, the composition $\pi^{-1} \circ F \circ \pi$, from points in the chart (v, y) to points in the same chart, is represented by

$$(v, y) \mapsto \left(\frac{P(v, 1)}{Q(v, 1)}, y^k Q(v, 1)\right) =: \left(\hat{f}(v), y^k Q(v, 1)\right).$$

That composition, from points in the chart (v, y) to points in the chart (x, u), is represented by $(v, y) \mapsto \left(y^k P(v, 1), (\hat{f}(v))^{-1}\right)$. Therefore, from the chart (v, y), the composition $\pi^{-1} \circ F \circ \pi$ is also analytically extended to the divisor $\mathbb{D}$.

As a consequence, we can conclude the existence and uniqueness of the blow-up $\tilde{F}$ of the mapping $F = (P, Q)$. Moreover, we show that the restriction of $\tilde{F}$ to the divisor $\mathbb{D}$, when we use the chart (x, u), is given by $f(u) = \frac{Q(1,u)}{P(1,u)}$.

2.3. Reformulation of Theorem 1

Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P, Q)$, be a map where $P, Q \in \mathbb{R}[x, y]$ are homogeneous polynomials of degree $k \geqslant 1$ which have no real lines in common. Let us note that its blow-up $\tilde{F}\colon \mathbb{M} \to \mathbb{M}$ enables us to study the dynamics of F, and also describe how F moves the lines in $\mathbb{R}^2$ passing through the origin.

On one hand, the blow-up $\tilde{F}$ outside the divisor $\mathbb{D}$ is equal to the composition $\pi^{-1} \circ F \circ \pi$; that is, $\tilde{F}$ outside $\mathbb{D}$ is equivalent to the mapping F outside the origin. On the other hand, the restriction of the blow-up $\tilde{F}$ to the divisor $\mathbb{D}$, denoted by $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, for simplicity, enables us to study how the slopes of the lines in $\mathbb{R}^2$ passing through the origin are moved by F. In particular, this information enables us to understand how the points in the basin of attraction of the origin are moved around this fixed point.

It is worthwhile to point that the fixed points of $F_{\mathbb{D}}$ correspond to the slopes of the lines through the origin which are invariant subsets of F. In Corollary 1, we show that the maximal number of fixed points of $F_{\mathbb{D}}$ is $k+1$, when $k \geqslant 2$.

Definition 2. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P, Q)$, be a map where $P, Q \in \mathbb{R}[x, y]$ are homogeneous polynomials of degree k, which have no real lines in common. The family $\mathcal{F}^k$ consists of all such mappings F having exactly $k+1$ invariant lines passing through the origin. That is, the maps F such that $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor, has exactly $k+1$ fixed points.

The subfamily $\mathcal{H}^k$ of $\mathcal{F}^k$ consists of all maps $F \in \mathcal{F}^k$ such that $F_{\mathbb{D}}$ has only hyperbolic fixed points (i. e., their derivatives are not 0, ± 1).

In Section 3, in Proposition 1, we prove that the family $\mathcal{F}^k$ is a real analytic variety. Also we show that $\mathcal{H}^k$ turns out to be an open subset of $\mathcal{F}^k$.

In Definition 3 we introduce the family $\mathcal{U}$ mentioned in Theorem 1, and after that, we present the reformulation of this theorem through Theorem 2.

Definition 3. Let $\mathcal{U}$ be the subfamily of $\mathcal{H}^2$ consisting of maps $F \in \mathcal{H}^2$, such that $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor, has exactly two attracting fixed points.

Theorem 2. *The family $\mathcal{U}$ is an open subset of $\mathcal{H}^2$. Each map $F \in \mathcal{U}$ satisfies that the third fixed point $\mathsf{p} \in \mathbb{D}$ of $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor, turns out to be a repeller. Moreover,*

(i) *there exists a unique point $\mathsf{q} \in \mathbb{D}$, $\mathsf{q} \neq \mathsf{p}$, such that $F_{\mathbb{D}}(\mathsf{q}) = \mathsf{p}$;*

(ii) *the punctured divisor $\mathbb{D} \setminus \{\mathsf{p}, \mathsf{q}\}$ consists of two segments, which are invariant sets under $F_{\mathbb{D}}$;*

(iii) *each attractor of $F_{\mathbb{D}}$ lies in one of the segments of $\mathbb{D} \setminus \{\mathsf{p}, \mathsf{q}\}$, which turns out to be its basin of attraction.*

In Example 1 we present a mapping $F \in \mathcal{H}^2$ such that $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor, has three repelling fixed points (that is, $F \notin \mathcal{U}$), and has chaotic dynamics in the sense of Devaney.

By Remark 2, Theorem 1 is an immediate consequence of Theorem 2, Proposition 1, and Example 1. Accordingly, the rest of the work is dedicated to their developments and their proofs. Proposition 1 is proved in Subsection 3.1, while Example 1 is developed in Subsection 3.2. The proof of Theorem 2 is divided into two parts: the first one is obtained in Subsection 3.2, and the second one is developed in Section 4.

3. Properties of the Maps in $\mathcal{F}^k$

In this section we present properties of $\mathcal{F}^k$, the family introduced in Definition 2. These properties will be used in the rest of the work. In particular, in Subsection 3.1 we prove Proposition 1 about the variety structures of the families $\mathcal{F}^k$ and $\mathcal{H}^k$.

In Subsection 3.2 we focus on the mappings F in the family $\mathcal{H}^2$; in Lemma 2, we describe the types of fixed points of $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of the blow-up of F to the divisor. Finally, we give an example where $F_{\mathbb{D}}$ has three repelling fixed points, and shows chaotic dynamics in the sense of Devaney (Example 1).

3.1. Some Properties of the Family $\mathcal{F}^k$

Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P, Q)$, be a map, with $P, Q \in \mathbb{R}[x, y]$ homogeneous polynomials of degree k, which have no real lines in common. We consider $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor. In Lemma 1 we obtain some properties of the map $F_{\mathbb{D}}$ which, in particular, enables us to prove Proposition 1 about the analytic variety structure of the family $\mathcal{F}^k$.

Recall that the charts on the divisor $\mathbb{D}$ are u, v, with $v = u^{-1}$. The map $F_{\mathbb{D}}$ has different expressions depending on the chart where it is represented:

$$f(u) := \frac{Q(1,u)}{P(1,u)}, \qquad \hat{f}(v) := \frac{P(v,1)}{Q(v,1)}, \tag{1}$$

where $f(u)$ represents $F_{\mathbb{D}}$ in the chart u into itself, and $\hat{f}(v)$ represents $F_{\mathbb{D}}$ in the chart v into itself. As a consequence, $\hat{f}(v)$ is the extension of $\big(f(1/v)\big)^{-1}$ (Subsection 2.2). We introduce Definition 4 in order to study the nature of the fixed points of the map $F_{\mathbb{D}}$ using the charts u, v of the divisor $\mathbb{D}$.

Definition 4. Let $P, Q \in \mathbb{R}[x,y]$ be homogeneous polynomials of degree k, which have no real lines in common. We consider $R := xQ - yP$, which is a homogeneous polynomial of degree $k+1$. Let $\alpha \in \mathbb{D}$ be the slope of a line in $\mathbb{R}^2$ passing through the origin, which is a simple linear factor of R. Then we define $\tau(\alpha)$ as

$$\frac{P(1,u_0)}{R_y(1,u_0)}, \qquad \text{or, alternatively,} \qquad -\frac{Q(v_0,1)}{R_x(v_0,1)}$$

(where $R_x := \frac{\partial R}{\partial x}$, $R_y := \frac{\partial R}{\partial y}$), depending on whether α corresponds to u_0 in the chart u, or α corresponds to v_0 in the chart v.

Lemma 1. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P,Q)$, be a map, with $P, Q \in \mathbb{R}[x,y]$ homogeneous polynomials of degree k, which have no real lines in common. We consider $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor. Then the fixed points of $F_{\mathbb{D}}$ are the slopes of the real linear factors of the homogeneous polynomial $R = xQ - yP$.*

If in addition the homogeneous polynomial R is the product of $k+1$ pairwise distinct real linear factors, then the slopes of its linear factors, $\mathsf{p}_1, \mathsf{p}_2, \dots, \mathsf{p}_{k+1} \in \mathbb{D}$, are the unique fixed points of the map $F_{\mathbb{D}}$. Moreover, their derivatives $F'_{\mathbb{D}}(\mathsf{p}_i)$ are expressed in terms of the constants $\tau(\mathsf{p}_i)$ given in Definition 4, which turn out to be different from zero:

$$F'_{\mathbb{D}}(\mathsf{p}_i) = 1 + \frac{1}{\tau(\mathsf{p}_i)},$$

whose sum satisfies $\tau(\mathsf{p}_1) + \tau(\mathsf{p}_2) + \cdots + \tau(\mathsf{p}_{k+1}) = -1$.

Proof. Using the expressions $f(u)$, $\hat{f}(v)$ of $F_{\mathbb{D}}$ in the different charts u, v given in (1), we immediately obtain the following equivalences:

$$f(u) = u \quad \Leftrightarrow \quad R(1,u) = 0, \qquad \hat{f}(v) = v \quad \Leftrightarrow \quad R(v,1) = 0,$$

since R and P have no common real linear factors different from x, and R and Q have no common real linear factors different from y. Therefore, the fixed points of $F_{\mathbb{D}}$ are the slopes of the real linear factors of the homogeneous polynomial R.

We analise the derivative of $F_{\mathbb{D}}$ at its fixed points. For that purpose, note that $f(u) = u + R(1,u)/P(1,u)$, while $\hat{f}(v) = v - R(v,1)/Q(v,1)$. As a consequence, the derivative of $f(u)$ at a fixed point u_0, and the derivative of $\hat{f}(v)$ at a fixed point v_0, are

$$f'(u_0) = 1 + \frac{R_y(1,u_0)}{P(1,u_0)}, \qquad \hat{f}'(v_0) = 1 - \frac{R_x(v_0,1)}{Q(v_0,1)}. \tag{2}$$

If we assume in addition that the polynomial R is the product of $k+1$ pairwise distinct real linear factors, we conclude that the slopes of these $k+1$ lines, denoted by $\mathsf{p}_1, \mathsf{p}_2, \dots, \mathsf{p}_{k+1} \in \mathbb{D}$, are the only fixed points of the map $F_{\mathbb{D}}$. By (2), the derivatives $F'_{\mathbb{D}}(\mathsf{p}_i)$ are expressed in terms of the constants $\tau(\mathsf{p}_i)$, $F'_{\mathbb{D}}(\mathsf{p}_i) = 1 + 1/\tau(\mathsf{p}_i)$, exactly as in the statement of this result.

By Residue Theorem ([2, Chapter V, §2]) applied to $P(1,u)/R(1,u)$, a rational function in $\mathbb{C}$, the sum of the residues of $P(1,u)/R(1,u)$, at all its poles in $\mathbb{C}$ is equal to the integral $\frac{1}{2\pi i}\int_\gamma \frac{P(1,u)}{R(1,u)}\mathrm{d}u$, where γ is a circle in $\mathbb{C}$ which traverses once in the counterclockwise direction, and such that the poles lie inside the open disc determined by γ.

By Definition 4, the sum of the residues of $P(1,u)/R(1,u)$ at all its poles is equal to $\tau(\mathsf{p}_1) + \tau(\mathsf{p}_2) + \cdots + \tau(\mathsf{p}_{k+1})$, if x is not a factor of R; on the other hand, if x is a factor of R and is represented as p_{k+1} in the divisor $\mathbb{D}$, this sum of the residues is equal to $\tau(\mathsf{p}_1) + \cdots + \tau(\mathsf{p}_k)$.

One can verify that the integral $\frac{1}{2\pi i}\int_\gamma \frac{P(1,u)}{R(1,u)}\mathrm{d}u$ is equal to the residue at zero of the rational function

$$\frac{1}{v^2} \cdot \left.\frac{P}{R}\right|_{(1,1/v)} . \tag{3}$$

Since R is equal to $xQ - yP$, the residue at zero of the function (3) is equal to -1, in the case where x is not a factor of R. If x is a factor of R and is represented as p_{k+1} in the divisor $\mathbb{D}$, then the residue at zero of the function (3) is equal to $-1-\tau(\mathsf{p}_{k+1})$. These properties enable us to conclude that the sum $\sum_{i=1}^{k+1}\tau(\mathsf{p}_i)$ is equal to -1. This completes the proof of the result. □

In the rest of this subsection we present some consequences of Lemma 1 related to the families $\mathcal{F}^k$ and $\mathcal{H}^k$, introduced in Definition 2.

Corollary 1. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F = (P,Q)$, be a map, with $P, Q \in \mathbb{R}[x,y]$ homogeneous polynomials of degree $k \geqslant 2$, which have no real lines in common. Then $R = xQ - yP$ is a nonzero homogeneous polynomial, and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of the blow-up of F to the divisor, has at most $k+1$ fixed points.*

Furthermore, $F_{\mathbb{D}}$ has exactly $k+1$ fixed points (i. e., $F \in \mathcal{F}^k$) if and only if R is the product of $k+1$ pairwise distinct real linear factors. Moreover, these fixed points are hyperbolic (i. e., $F \in \mathcal{H}^k$) if and only if the constants $\tau(\mathsf{p}) \neq 0$ associated with the slopes $\mathsf{p} \in \mathbb{D}$ of the linear factors of R are different from -1, $-1/2$.

Remark 3. Given $\alpha \in \mathbb{RP}^1$, we denote by P_α the linear polynomial $y - ax$, if α corresponds to a in the chart u, or x if α corresponds to zero in the chart v. We consider $R \in \mathbb{R}[x,y]$ a homogeneous polynomial of degree $k+1$, which is the product of $k+1$ pairwise distinct real linear factors, $R = cP_{\alpha_1}\cdots P_{\alpha_{k+1}}$; with each factor P_{α_i} is associated a real number $\tau_i \neq 0$, such that their sum $\sum_{i=1}^{k+1}\tau_i$ is equal to -1. Then, the polynomial $P(1,u) \in \mathbb{R}[u]$ is defined as

$$R(1,u)\sum_{i=1}^{k}\frac{\tau_i}{P_{\alpha_i}(1,u)}\,, \quad \text{or, alternatively,} \quad R(1,u)\sum_{i=1}^{k+1}\frac{\tau_i}{P_{\alpha_i}(1,u)}\,,$$

depending on whether $x = P_{\alpha_{k+1}}$ is, or it is not, a factor of R. In this way, we obtain two homogeneous polynomials $P := x^k P(1, y/x)$ and $Q \in \mathbb{R}[x,y]$ such that xQ is equal to $R + yP$. One can check directly that the pair $F = (P,Q)$ belongs to the family $\mathcal{F}^k$.

By Lemma 1, we immediately conclude that $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{D}$ are the fixed points of $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$, the restriction of the blow-up of F to the divisor. Moreover, their derivatives $F'_{\mathbb{D}}(\alpha_i)$ are equal to $1 + 1/\tau_i$, for $i = 1, \dots, k+1$.

Proposition 1. *The family of maps $\mathcal{F}^k$ is a real analytic variety, and its subfamily $\mathcal{H}^k$ constitutes an open subset of $\mathcal{F}^k$.*

Proof. We denote by $\tilde{\mathcal{F}}^k$ the family of all homogeneous polynomials $R \in \mathbb{R}[x, y]$ of degree $k+1$, each of them associated with a collection Γ of $k+1$ nonzero real numbers, satisfying the following properties: the polynomial R is the product of $k+1$ pairwise distinct real linear factors, and each factor is associated with a nonzero real number τ_i in the collection Γ, such that their sum $\sum_{i=1}^{k+1} \tau_i$ is equal to -1. By Remark 3, we obtain an element of the family $\mathcal{F}^k$ from an element of the family $\tilde{\mathcal{F}}^k$. Using Lemma 1, this establishes a one-to-one correspondence between the family $\mathcal{F}^k$ and the family $\tilde{\mathcal{F}}^k$.

We consider $\mathbb{D}^{k+1}$, the Cartesian product of the divisor $\mathbb{D}$. Let $\mathcal{A}^{k+1}$ be the open subset of $\mathbb{D}^{k+1}$ consists of $(\alpha_i) \in \mathbb{D}^{k+1}$ having pairwise distinct coordinates: $\alpha_i \neq \alpha_j$, if $i \neq j$. Let $\mathcal{B}^{k+1}$ be the subvariety of $(\mathbb{R}^*)^{k+1}$ consists of its elements (τ_i) such that the sum of its coordinates $\sum \tau_i$ is equal to -1. We consider the mapping from $\mathbb{R}^* \times \mathcal{A}^{k+1} \times \mathcal{B}^{k+1}$ to the family $\tilde{\mathcal{F}}^k$,

$$(c, (\alpha_i), (\tau_i)) \quad \longmapsto \quad R := cP_{\alpha_1} \cdots P_{\alpha_{k+1}}\,, \quad \tau_i \;\text{ associated with }\; P_{\alpha_i} \tag{4}$$

(here we use the notation introduced in Remark 3). This mapping is onto but not injective; in fact, two elements $(c, (\alpha_i), (\tau_i))$ and $(c', (\alpha'_i), (\tau'_i))$ have the same image if and only if $c = c'$, and there exists σ in S_{k+1}, the permutation group of $k+1$ elements, such that $(\alpha'_i) = (\alpha_{\sigma(i)})$, and $(\tau'_i) = (\tau_{\sigma(i)})$.

Denoting by $\mathcal{C}^{k+1}$ the quotient space $\big(\mathcal{A}^{k+1} \times \mathcal{B}^{k+1}\big)/S_{k+1}$, we can conclude that the mapping (4) factors through $\mathbb{R}^* \times \mathcal{C}^{k+1}$. In this way, we obtain a one-to-one correspondence between the analytic real variety $\mathbb{R}^* \times \mathcal{C}^{k+1}$ and the family $\tilde{\mathcal{F}}^k$.

Using the one-to-one correspondences defined between the families $\mathcal{F}^k$ and $\tilde{\mathcal{F}}^k$, and between $\tilde{\mathcal{F}}^k$ and the space $\mathbb{R}^* \times \mathcal{C}^{k+1}$, we endow the structure of real analytic variety to the family $\mathcal{F}^k$. Moreover, through these correspondences and the canonical projection $\mathbb{R}^* \times \mathcal{A}^{k+1} \times \mathcal{B}^{k+1} \twoheadrightarrow \mathbb{R}^* \times \mathcal{C}^{k+1}$, the open subsets of the product $\mathbb{R}^* \times \mathcal{A}^{k+1} \times \mathcal{B}^{k+1}$ are sent to open subsets on the space $\mathbb{R}^* \times \mathcal{C}^{k+1} \simeq \mathcal{F}^k$. In particular, we have that the subfamily $\mathcal{H}^k \subseteq \mathcal{F}^k$ is an open subset of $\mathcal{F}^k$, since this subfamily comes from the elements $(c, (\alpha_i), (\tau_i))$ of $\mathbb{R}^* \times \mathcal{A}^{k+1} \times \mathcal{B}^{k+1}$, such that $\tau_i \neq -1, -1/2$ (Corollary 1). The proof is complete. □

3.2. The Fixed Points of $F_{\mathbb{D}}$, for $F \in \mathcal{H}^2$

In this subsection we focus on the family $\mathcal{H}^2$ introduced in Definition 2. In Lemma 2 we list the types of fixed points that $F_{\mathbb{D}}$ can have, where $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$ is the restriction of the blow-up of $F \in \mathcal{H}^2$ to the divisor. Among these types is found the subfamily of maps $F \in \mathcal{H}^2$ such that $F_{\mathbb{D}}$ has exactly two attracting fixed points, i. e., the subfamily $\mathcal{U}$ given in Definition 3. In Remark 4, we note that $\mathcal{U}$ constitutes an open subset of $\mathcal{H}^2$.

On the other hand, there are mappings $F \in \mathcal{H}^2$ where $F_{\mathbb{D}}$ has three repelling fixed points. In Example 1 we give a map with this property. It turns out that this map shows chaotic dynamics in the sense of Devaney.

Definition 5. Let F be a map in the family $\mathcal{H}^2$. As before, $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$ is the restriction of its blow-up to the divisor. Given a fixed point $\mathsf{p} \in \mathbb{D}$ of $F_{\mathbb{D}}$, we consider its derivative $\lambda := F'_{\mathbb{D}}(\mathsf{p})$. The fixed point $\mathsf{p} \in \mathbb{D}$ will be called *positive attractor* if $0 < \lambda < 1$, and *negative attractor* if $-1 < \lambda < 0$. Furthermore, p will be called *positive repeller* if $\lambda > 1$, and *negative repeller* if $\lambda < -1$.

In what follows, we use $\mathfrak{a}_+$ and $\mathfrak{a}_-$ to denote that an attracting fixed point of the mapping $F_{\mathbb{D}}$ is positive or negative, respectively. We use $\mathfrak{r}_+$ and $\mathfrak{r}_-$ to denote that a repelling fixed point is positive or negative, respectively.

Lemma 2. *Let $F \in \mathcal{H}^2$ and $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. Then,*

(i) *If $F_{\mathbb{D}}$ has two attractors, then the third point must be a positive repeller. Furthermore, the only possible combinations are*

$$\mathfrak{a}_+, \mathfrak{a}_+, \mathfrak{r}_+\,, \qquad \mathfrak{a}_+, \mathfrak{a}_-, \mathfrak{r}_+\,, \qquad \mathfrak{a}_-, \mathfrak{a}_-, \mathfrak{r}_+\,.$$

(ii) *If $F_{\mathbb{D}}$ has only one attractor, then the only possible combinations are*

$$\mathfrak{a}_+, \mathfrak{r}_+, \mathfrak{r}_+\,, \qquad \mathfrak{a}_+, \mathfrak{r}_+, \mathfrak{r}_-\,, \qquad \mathfrak{a}_-, \mathfrak{r}_+, \mathfrak{r}_-\,, \qquad \mathfrak{a}_-, \mathfrak{r}_-, \mathfrak{r}_-\,.$$

(iii) *The only possible combination when the map $F_{\mathbb{D}}$ has no attractors, is that it has three negative repellers $\mathfrak{r}_-, \mathfrak{r}_-, \mathfrak{r}_-$.*

All combinations listed as possible, are realizable by the fixed points of a map $F_{\mathbb{D}}$, for some $F \in \mathcal{H}^2$.

Proof. We denote by p_1, p_2, $\mathsf{p}_3 \in \mathbb{D}$ the three fixed points of the map $F_{\mathbb{D}}$, and by $\lambda_i = 1 + 1/\tau_i$ the derivative of $F_{\mathbb{D}}$ at the point p_i. By Lemma 1 the nonzero real numbers τ_i satisfy

$$\tau_1 + \tau_2 + \tau_3 = -1 \qquad \Longleftrightarrow \qquad \tau_3 = -\big(1 + \tau_1 + \tau_2\big)\,.$$

First, we assume that the mapping $F_{\mathbb{D}}$ has exactly two attractors. Without loss of generality we assume that p_1, p_2 are such attractors. We note the following:

— Both fixed points p_1 and p_2 are $\mathfrak{a}_+$ if and only if τ_1, τ_2 are in $(-\infty, -1)$. This implies that $\tau_3 \in (1, \infty)$, and as a consequence, the point p_3 is $\mathfrak{r}_+$.

— Both fixed points p_1 and p_2 are $\mathfrak{a}_-$ if and only if τ_1, τ_2 are in $(-1, -1/2)$. This implies that $\tau_3 \in (0, 1)$, and as a consequence, the point p_3 is $\mathfrak{r}_+$.

— The point p_1 is $\mathfrak{a}_+$, and the point p_2 is $\mathfrak{a}_-$ if and only if $\tau_1 \in (-\infty, -1)$, and $\tau_2 \in (-1, -1/2)$. This implies that $\tau_3 \in (1/2, \infty)$, and as a consequence, the point p_3 is $\mathfrak{r}_+$.

Now we assume that the map $F_{\mathbb{D}}$ has exactly two repellers, and without loss of generality we assume that p_1, p_2 are such repellers. Then,

— Both fixed points p_1 and p_2 are $\mathfrak{r}_+$ if and only if τ_1, τ_2 are in $(0, \infty)$. This implies that $\tau_3 \in (-\infty, -1)$, and as a consequence, the point p_3 is $\mathfrak{a}_+$.

— Both fixed points p_1 and p_2 are $\mathfrak{r}_-$ if and only if τ_1, τ_2 are in $(-1/2, 0)$. This implies that $\tau_3 \in (-1, 0)$. Since p_3 is hyperbolic, then $\tau_3 \neq -1/2$, and as a consequence, the point p_3 is either $\mathfrak{a}_-$ or $\mathfrak{r}_-$.

— The point p_1 is $\mathfrak{r}_+$, and the point p_2 is $\mathfrak{r}_-$ if and only if $\tau_1 \in (0, \infty)$, and $\tau_2 \in (-1/2, 0)$. This implies that $\tau_3 \in (-\infty, -1/2)$. Since p_3 is hyperbolic, then $\tau_3 \neq -1$, and as a consequence, the point p_3 is either $\mathfrak{a}_+$ or $\mathfrak{a}_-$.

So far we complete the proof of the items 1, 2 and 3 of the statement. Finally, we note that in each case we have considered, any element of the interval to which τ_3 belongs, can be realized as $-(1+\tau_1+\tau_2)$, choosing τ_1, τ_2 appropriately in the respective intervals. Therefore, using Remark 3, we conclude that all combinations listed as possible, are realizable by the fixed points of a mapping $F_{\mathbb{D}}$, for some $F \in \mathcal{H}^2$. □

Remark 4. By fixing any combination listed in Lemma 2, we obtain an open subset of the family $\mathcal{H}^2$, considering all the maps $F \in \mathcal{H}^2$ such that $F_{\mathbb{D}}$ has fixed points satisfying this combination.

In order to prove the claim, we consider the variety structure of $\mathcal{F}^2$ described in the proof of Proposition 1, where we proved that the family $\mathcal{H}^2$ is an open subset of $\mathcal{F}^2$. In this setting, the canonical projection $\mathbb{R}^* \times \mathcal{A}^3 \times \mathcal{B}^3 \twoheadrightarrow \mathbb{R}^* \times \mathcal{C}^3 \simeq \mathcal{F}^2$ is an open analytic map.

We consider the open subset of $\mathbb{R}^* \times \mathcal{A}^3 \times \mathcal{B}^3$, which consists of the points $(c, (\mathsf{p}_i), (\tau_i))$ such that τ_1, τ_2, τ_3 belong to the open intervals corresponding to the given combination. Then its image under the projection turns out to be an open subset of $\mathcal{F}^2$, namely, the mappings $F \in \mathcal{H}^2$ such that $F_{\mathbb{D}}$ has fixed points satisfying the given combination.

Example 1. [Three repelling fixed points]. Let us consider the map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F = \big(P, Q\big) = (x^2 - y^2, -2xy)$. If we identify the plane $\mathbb{R}^2$ with the space of complex numbers $\mathbb{C}$, the map F corresponds with the mapping $z \to \overline{z^2}$. In the following we show that the map F is a member of the family $\mathcal{H}^2$. Furthermore, we show that the restriction of the blow-up of F to the divisor, $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, has three negative repelling fixed points. Also we show that the function F, restricted to a subset of $\mathbb{R}^2$, is chaotic in the sense of Devaney, and how this property is inherited to the map $F_{\mathbb{D}}$.

We begin by showing that the map $F_{\mathbb{D}}$ has exactly three fixed points in $\mathbb{D}$. Also it is not difficult to show that the derivative of $F_{\mathbb{D}}$ in each of these points is -2. Since the polynomials P and Q have no real lines in common, the slopes of the real lines that divide the polynomial

$$R := xQ - yP = -2x^2y - x^2y + y^3 = y\big(y - \sqrt{3}x\big)\big(y + \sqrt{3}x\big),$$

are the fixed points of the map $F_{\mathbb{D}}$ (Lemma 1). These three points correspond to $u_1 = -\sqrt{3}$, $u_2 = 0$ and $u_3 = \sqrt{3}$ in the chart u. The derivatives are

$$\lambda_i = 1 + \frac{1}{\tau_i}, \qquad \text{with} \quad \tau_i = \frac{P(1,u_i)}{R_y(1,u_i)}, \qquad i = 1,2,3$$

respectively (Lemma 1). Since $P(1,u) = 1 - u^2$, and $R_y(1,u) = \Sigma_{i=1}^3 \Pi_{j\neq i}(u - u_j)$, we have that

$$\tau_1 = \frac{1-3}{-\sqrt{3}(-2\sqrt{3})} = -\frac{1}{3}, \qquad \tau_2 = -\frac{1}{3}, \qquad \tau_3 = \frac{1-3}{\sqrt{3}(2\sqrt{3})} = -\frac{1}{3}.$$

Hence, $\lambda_i = 1 - 3 = -2$, for $i = 1,2,3$. It follows that F belongs to the family $\mathcal{H}^2$.

Let us recall the notion of chaotic map due to Devaney (see [3, Definition 8.5]). Let $X = (X, d)$ be a compact metric space. Let $\varphi\colon X \to X$ be a continuous map. We say that the map φ is *transitive* provided that for every pair of nonempty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We say that φ is *chaotic* provided that the set of periodic points of φ is dense in X, and φ is transitive.

Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ given by $\varphi(z) = z^2$. It is known that φ restricted to the unit circle $\mathbb{S}^1$ is chaotic (see [3, Example 8.6]). It is not difficult to show that $\varphi^2 = \varphi \circ \varphi$ is chaotic on $\mathbb{S}^1$ as well. Note that the circle $\mathbb{S}^1$ is an invariant set under F. Since for every $z \in \mathbb{C}$, $F^2(z) = \varphi^2(z) = z^4$, it follows that F^2 and F are chaotic maps restricted to the circle $\mathbb{S}^1$.

Let $h\colon \mathbb{S}^1 \to \mathbb{D}$ be the map defined in the following way: let $(x, y) \in \mathbb{S}^1$. If $x \neq 0$, let $h(x, y) \in \mathbb{D}$ be the corresponding point y/x in the chart u; if $y \neq 0$, let $h(x, y) \in \mathbb{D}$ be the corresponding point x/y in the chart v. Notice that h is well defined in the divisor $\mathbb{D}$, since we have that $v = u^{-1}$. The mapping h is continuous, furthermore it is analytic.

Now our goal is to show that $h : \mathbb{S}^1 \to \mathbb{D}$ is an onto map. Let $r \in \mathbb{R}$, and let $p \in \mathbb{D}$ be the point corresponding to r in the chart u. Let $x = 1/\sqrt{1 + r^2}$ and $y = r/\sqrt{1 + r^2}$. Note that $(x, y) \in \mathbb{S}^1$ and $h(x, y) = r$, in the chart u. There is only just one point that is not covered by the chart u, say p. Since $p = h(0, 1) = h(0, -1)$, the map h is onto. Furthermore, it is not difficult to show, from the definition of h, that each point of $\mathbb{D}$ has exactly two preimages under h.

The map $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, in the chart u, is given by $f(u) = -2u/(1-u^2)$. Let $(x, y) \in \mathbb{S}^1$. Using the chart u we have the following:

$$h \circ F|_{\mathbb{S}^1}(x, y) = h(x^2 - y^2, -2xy) = \frac{-2xy}{x^2 - y^2},$$

and

$$F_{\mathbb{D}} \circ h(x, y) = F_{\mathbb{D}}\left(\frac{y}{x}\right) = \frac{-2(y/x)}{1 - (y/x)^2} = \frac{-2xy}{x^2 - y^2}.$$

We obtain a similar result using the chart v. Hence $h \circ F|_{\mathbb{S}^1} = F_{\mathbb{D}} \circ h$. Now it readily follows that $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ is a chaotic mapping.

Remark 5. The *entropy* is another important concept in the study of dynamical properties of the maps $\varphi\colon X \to X$. The value of the entropy of φ, $\operatorname{ent}(\varphi)$, has only three options: 0, positive, or ∞. According to Block and Coppel the map φ is chaotic provided that $\operatorname{ent}(\varphi)$ is positive or ∞. For definition and basic properties of the entropy we refer the reader to [4–6].

It is known that the entropy of the map $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(z) = z^2$, is $\log(2)$ (see [6, Proposition 1.1]). It follows that $2\log(2) = \operatorname{ent}(\varphi^2) = \operatorname{ent}(F_{\mathbb{D}}^2) = 2\operatorname{ent}(F_{\mathbb{D}})$. Hence, $\operatorname{ent}(F_{\mathbb{D}}) = \log(2) > 0$.

4. Proof of Theorem 2

Let us consider maps $F \in \mathcal{U}$, that is, maps $F \in \mathcal{H}^2$ such that $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$, the restriction of its blow-up to the divisor, has exactly two attracting fixed points. This family constitutes an open subset of $\mathcal{H}^2$ (Remark 4), where each mapping F satisfies that the third fixed point $\mathsf{p} \in \mathbb{D}$ of $F_{\mathbb{D}}$ turns out to be a positive repeller (Lemma 2). Given these properties, Theorem 2 will be completed if we prove the following:

- there exists a unique point $\mathsf{q} \in \mathbb{D}$, $\mathsf{q} \neq \mathsf{p}$, such that $F_{\mathbb{D}}(\mathsf{q}) = \mathsf{p}$;
- the punctured divisor $\mathbb{D} \setminus \{\mathsf{p}, \mathsf{q}\}$ consists of two segments, which are invariant sets under $F_{\mathbb{D}}$;
- each attractor of $F_{\mathbb{D}}$ lies in one of the segments of $\mathbb{D} \setminus \{\mathsf{p}, \mathsf{q}\}$, which turns out to be its basin of attraction.

We begin by showing that these properties follow from Theorem 7, which is given in terms of Definition 6.

Definition 6. Let us consider maps $F = (P, Q) \in \mathcal{H}^2$ such that the homogeneous polynomial $R = xQ - yP$ is equal to $xy(y - x)$. We denote by p_0, p_1, $\mathsf{p}_\infty \in \mathbb{D}$ the respective slopes of the lines $y = 0$, $y - x = 0$, $x = 0$ (they correspond to 0, 1, $\pm\infty$ in the chart u, respectively). We define three subfamilies $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{A}_\pm$ of such maps: the subfamily $\mathcal{A}_+$ consists of all maps F such that $F_{\mathbb{D}}$ has p_0, p_∞ as positive attractors; the subfamily $\mathcal{A}_-$ consists of all maps F such that $F_{\mathbb{D}}$ has p_0, p_∞ as negative attractors; finally, the subfamily $\mathcal{A}_\pm$ consists of all maps F such that $F_{\mathbb{D}}$ has p_0, p_∞ as positive and negative attractors, respectively. The union $\mathcal{A}_+ \cup \mathcal{A}_- \cup \mathcal{A}_\pm$ is denoted by $\mathcal{A}$.

Lemma 3. *Let $F \in \mathcal{H}^2$ and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. If $F_{\mathbb{D}}$ has two attracting fixed points, then F is equivalent to a map in the family $\mathcal{A}$, by means of a linear conjugation.*

Proof. Let α_0, α_1, $\alpha_\infty \in \mathbb{D}$ be the fixed points of $F_{\mathbb{D}}$, such that α_0, α_∞ are both attractors. In the case where the attractors have different signs, we assume that α_0 and α_∞ are positive and negative attractors, respectively. We consider $\mathcal{L}_i$ the line in $\mathbb{R}^2$ passing through the origin, which has the slope α_i, for $i = 0, 1, \infty$.

There exists a linear transformation which sends $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_\infty$ into the lines $y = 0$, $y - x = 0$, $x = 0$, respectively. Then, except for the conjugation by this linear transformation, the homogeneous polynomial R is equal to $cxy(y - x)$, $c \neq 0$, and hence, the slopes p_0, p_1, p_∞ are the fixed points of $F_{\mathbb{D}}$ (Lemma 1). Moreover, the attractors are p_0 and p_∞, which are positive and negative, respectively, in the case where they have different signs. Finally, the coefficient c of R becomes 1, using a homothetic transformation as conjugation. □

Lemma 3 enables us to conclude that the second part of Theorem 2 is an immediate consequence of Theorem 7.

Theorem 7. *Let $F \in \mathcal{A}$ and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. Then*

(i) *there exists a unique point $\mathsf{q}_1 \in \mathbb{D}$, $\mathsf{q}_1 \neq \mathsf{p}_1$, such that $F_{\mathbb{D}}(\mathsf{q}_1) = \mathsf{p}_1$;*

(ii) *the punctured divisor $\mathbb{D} \setminus \{\mathsf{p}_1, \mathsf{q}_1\}$ consists of two segments, which are invariant sets under $F_{\mathbb{D}}$;*

(iii) *each attractor of $F_{\mathbb{D}}$ lies in one of the segments of $\mathbb{D} \setminus \{\mathsf{p}_1, \mathsf{q}_1\}$, which turns out to be its basin of attraction.*

The proof of Theorem 7 is divided in three parts. The first one, given in Subsection 4.1, enables us to locate in $\mathbb{D}$ some relevant points of the mapping $F_{\mathbb{D}}$, among them the point $\mathsf{q}_1 \in \mathbb{D}$. The second part, given in Subsection 4.2, is devoted to prove the invariance under $F_{\mathbb{D}}$ of the connected components of the punctured divisor $\mathbb{D} \setminus \{\mathsf{p}_1, \mathsf{q}_1\}$. In the third and last part, given in Subsection 4.3, we deal with the problem of the basins of attraction.

4.1. The Critical Points of $F_{\mathbb{D}}$

Let us consider F a map in the family $\mathcal{A}$, and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ the restriction of its blow-up to the divisor. Recall that the mappings $f(u)$ and $\hat{f}(v)$ given in (1), represent $F_{\mathbb{D}}$ from the chart u to itself and from the chart v to itself, respectively. Every point in the divisor $\mathbb{D}$, viewed from the charts, is in the domain of $f(u)$, or in the domain of $\hat{f}(v)$.

If $\alpha \in \mathbb{D}$ is not a critical point of $F_{\mathbb{D}}$, we define *the sign of the derivative of $F_{\mathbb{D}}$ at α*, as the sign that $f'(u)$ or $\hat{f}'(v)$ takes at the point α expressed in the corresponding chart.

Lemma 4 enables us to locate the critical points of the map $F_{\mathbb{D}}$, and to learn the sign of the derivative of $F_{\mathbb{D}}$ at the points which are not critical (Fig. 1). Its statement uses the notation introduced in Definition 1.

Lemma 4. *Let $F \in \mathcal{A}$ and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. Then the map $F_{\mathbb{D}}$ has two different critical points α_0, $\alpha_\infty \in \mathbb{D}$. Furthermore, there exists a unique point $\mathsf{q}_\infty \in \mathbb{D}$, $\mathsf{q}_\infty \neq \mathsf{p}_\infty$, such that $F_{\mathbb{D}}(\mathsf{q}_\infty) = \mathsf{p}_\infty$.*

The location of the three points α_0, α_∞, q_∞ on the divisor $\mathbb{D}$ is as follows:

(i) *If $F \in \mathcal{A}_+$, then α_0, $\alpha_\infty \in (\mathsf{p}_\infty, \mathsf{p}_0)$. Moreover, $\mathsf{q}_\infty \in (\alpha_\infty, \alpha_0)$.*

(ii) *If $F \in \mathcal{A}_-$, then $\alpha_0 \in (\mathsf{p}_0, \mathsf{p}_1)$ and $\alpha_\infty \in (\mathsf{p}_1, \mathsf{p}_\infty)$. Moreover, $\mathsf{q}_\infty \in (\mathsf{p}_1, \alpha_\infty)$.*

(iii) *If $F \in \mathcal{A}_\pm$, then $\alpha_0 \in (\mathsf{p}_\infty, \mathsf{p}_0)$ and $\alpha_\infty \in (\mathsf{p}_1, \mathsf{p}_\infty)$. Moreover, q_∞ is in $(\mathsf{p}_1, \alpha_\infty)$.*

Furthermore, the derivative of $F_{\mathbb{D}}$ has positive sign on the segment $(\alpha_0, \alpha_\infty)$ of the divisor $\mathbb{D}$, and it has negative sign on the segment $(\alpha_\infty, \alpha_0) \subseteq \mathbb{D}$.

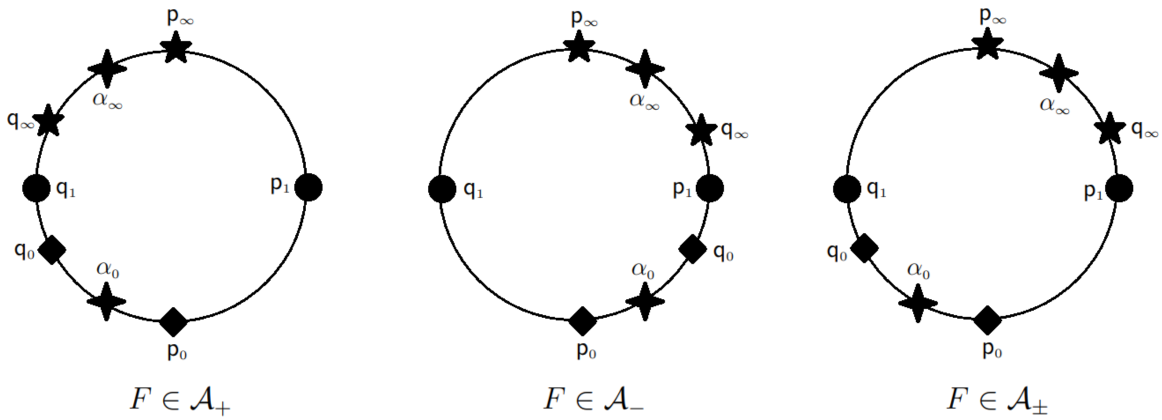

Fig. 1. The location of α_0, α_∞, q_∞ on the divisor $\mathbb{D}$

Proof. Since the map $F = (P, Q)$ is in the family $\mathcal{A}$ (see Definition 6), then the homogeneous polynomial $R = xQ - yP$ is equal to $xy(y - x)$. Using Remark 3, we conclude that

$$P(x,y) = x\big(\tau_0(y-x) + \tau_1 y\big), \quad xQ = R + yP, \tag{5}$$

where $\tau_i := \tau(\mathsf{p}_i)$, $i = 0, 1, \infty$, are the real numbers associated with p_0, p_1, $\mathsf{p}_\infty \in \mathbb{D}$, the fixed points of $F_{\mathbb{D}}$. Let us recall that $\tau_0, \tau_1, \tau_\infty$ are nonzero real numbers whose sum $\tau_0 + \tau_1 + \tau_\infty$ is equal to -1, and satisfy $F'_{\mathbb{D}}(\mathsf{p}_i) = 1 + 1/\tau_i$.

Recall that the mappings $f(u)$ and $\hat{f}(v)$ given in (1), represent $F_{\mathbb{D}}$ from the chart u to itself and from the chart v to itself, respectively. Therefore, $\hat{f}(v)$ is the extension of $\big(f(1/v)\big)^{-1}$. Since $F_{\mathbb{D}}$ is expressed in the charts u, v as the quotient of two polynomials of degree less or equal to 2, then the preimage $F_{\mathbb{D}}^{-1}(\alpha)$ of a given value $\alpha \in \mathbb{D}$, has at most two points.

Therefore, the point $\mathsf{q}_\infty \in \mathbb{D}$, which corresponds to the root of $P(1,u)$ in the chart u, is the unique point different to the fixed point p_∞ (which is seen as $\pm\infty$ from the chart u), such that $F_{\mathbb{D}}(\mathsf{q}_\infty) = \mathsf{p}_\infty$. The location of q_∞ on the divisor $\mathbb{D}$ can be obtained from the expression of $P(1,u)$ given in (5). Therefore, the mapping $f(u)$ represents, from the chart u, the map $F_{\mathbb{D}}$ sending $\mathbb{D} \setminus \{\mathsf{p}_\infty, \mathsf{q}_\infty\}$ into $\mathbb{D} \setminus \{\mathsf{p}_\infty\}$.

Considering the point $\mathsf{q}_\infty \in \mathbb{D}$ in the chart v, by a direct analysis of the mapping $\hat{f}(v)$, we can conclude that q_∞ is not a critical point of $F_{\mathbb{D}}$. As a consequence, by analysing the derivative of $f(u)$ and from the information given in (5), we obtain all the critical points of the map $F_{\mathbb{D}}$ and their location on the divisor $\mathbb{D}$.

Since the sign of $F'_{\mathbb{D}}$ does not depend on the choice of the chart where $F_{\mathbb{D}}$ is expressed, then the sign of $F'_{\mathbb{D}}$ is equal to the sign of $f'(u)$ at the points in the punctured divisor

$\mathbb{D} \setminus \{\mathsf{p}_\infty, \mathsf{q}_\infty\}$. By continuity in the derivatives, the sign of $F'_{\mathbb{D}}$ at q_∞ is equal to the sign of $f'(u)$ at the points sufficiently close to the expression of q_∞ in the chart u. □

Corollary 2 enables us to locate the points q_0, $\mathsf{q}_1 \in \mathbb{D}$, which are sent to the fixed points p_0, p_1, respectively. These points are also depicted in Fig. 1.

Corollary 2. *Let $F \in \mathcal{A}$ and $F_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. Then there exist two unique points q_0, $\mathsf{q}_1 \in \mathbb{D}$, $\mathsf{q}_i \neq \mathsf{p}_i$, such that $F_{\mathbb{D}}(\mathsf{q}_i) = \mathsf{p}_i$, for $i = 0, 1$. The location of the two points q_0, q_1 on the divisor $\mathbb{D}$ is as follows:*

(i) *if $F \in \mathcal{A}_+$, then $[\mathsf{q}_1, \mathsf{q}_0] \subseteq (\mathsf{q}_\infty, \alpha_0)$;*

(ii) *if $F \in \mathcal{A}_-$, then $\mathsf{q}_0 \in (\alpha_0, \mathsf{p}_1)$ and $\mathsf{q}_1 \in (\mathsf{p}_\infty, \mathsf{p}_0)$;*

(iii) *if $F \in \mathcal{A}_\pm$, then $[\mathsf{q}_1, \mathsf{q}_0] \subseteq (\mathsf{p}_\infty, \alpha_0)$.*

Proof. We have already noted that $F_{\mathbb{D}}$ is expressed in the charts u, v as the quotient of two polynomials of degree less or equal to 2, and as a result, the preimage $F_{\mathbb{D}}^{-1}(\alpha)$ of a given value $\alpha \in \mathbb{D}$, has at most two points.

Hence, if there exist $\mathsf{q}_i \in \mathbb{D}$ such that $\mathsf{q}_i \neq \mathsf{p}_i$, and $F_{\mathbb{D}}(\mathsf{q}_i) = \mathsf{p}_i$, for $i = 0, 1$, they would be the only points having these properties; and moreover, they would be in $\mathbb{D} \setminus \{\mathsf{p}_\infty, \mathsf{q}_\infty\}$. Then, it is sufficient to analyse the mapping $f(u)$ given in (1), since it represents, from the chart u, the map $F_{\mathbb{D}}$ sending $\mathbb{D} \setminus \{\mathsf{p}_\infty, \mathsf{q}_\infty\}$ into $\mathbb{D} \setminus \{\mathsf{p}_\infty\}$.

To simplify the notation, we identify each point in the divisor $\mathbb{D}$ with its expression in the chart u. First, we assume that F belongs to the family $\mathcal{A}_\pm$. Then, by Lemma 4, $\alpha_0 < \mathsf{p}_0 < \mathsf{p}_1 < \mathsf{q}_\infty$, and $f'(u)$ has negative sign in $(-\infty, \alpha_0)$, and positive sign in $(\alpha_0, \mathsf{q}_\infty)$. Therefore, $f(u)$ is strictly decreasing in $(-\infty, \alpha_0]$, and it is strictly increasing in $[\alpha_0, \mathsf{q}_\infty)$. This implies that $f(\alpha_0) < f(\mathsf{p}_0) = \mathsf{p}_0$, and moreover, the restriction of $f(u)$ to the interval $(-\infty, \alpha_0]$ is injective whith image $[f(\alpha_0), \infty)$. Therefore, there exist $\mathsf{q}_0, \mathsf{q}_1$ in the interval $(-\infty, \alpha_0]$, such that $f(\mathsf{q}_i) = \mathsf{p}_i$, for $i = 0, 1$; furthermore, they satisfy $\mathsf{q}_1 < \mathsf{q}_0$.

This proves the result for the family $\mathcal{A}_\pm$. The proof of the other cases is similar to that given for the family $\mathcal{A}_\pm$. □

4.2. Invariance of Segments of $\mathbb{D}$

Let us consider the maps F in the family $\mathcal{A}$. In this subsection, the main result is Proposition 2, which enables us to conclude the invariance under $F_{\mathbb{D}}$ of some segments of the divisor $\mathbb{D}$.

To prove Proposition 2 we need Lemma 8.5 in [7], which is formulated as Lemma 5 for the sake of completeness. For that purpose, we consider $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational function of degree $d \geqslant 2$ defined on the Riemman sphere $\hat{\mathbb{C}}$. We suppose that there exists $p \in \hat{\mathbb{C}}$ an attracting fixed point of h, with $\lambda := h'(p) \neq 0$. We denote by A its basin of attraction, and by A_0 its immediate basin of attraction (i. e., the connected component of A which contains p). Corollary 8.4 in [7] enables us to conclude that there exists a holomorphic map $\phi : A \to \mathbb{C}$ such that $\phi \circ h(x) = \lambda(\phi(x))$ for every $x \in A$; moreover, ϕ is a biholomorphism taking a neighborhood of p onto a neighborhood of zero. We consider a small neighborhood D_ε of zero, where there exists a holomorphic map $\psi_\varepsilon : D_\varepsilon \to A_0$ which is inverse to the map ϕ, that is, $\phi \circ \psi_\varepsilon(w) = w$, for every $w \in D_\varepsilon$, and $\psi_\varepsilon(0) = p$.

Lemma 5. [7, Lemma 8.5]. *This local inverse $\psi_\varepsilon : D_\varepsilon \to A_0$ extends, by analytic continuation, to some maximal open disk D_r about the origin in $\mathbb{C}$. This yields a uniquely defined holomorphic map $\psi : D_r \to A_0$ with $\psi(0) = p$ and $\phi \circ \psi(w) = w$. Furthermore, ψ extends homeomorphically over the boundary circle ∂D_r, and the image $\psi(\partial D_r)$ necessarily contains a critical point of h.*

Corollary 3. *Let $F \in \mathcal{A}$ and $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. We consider the critical points α_0, $\alpha_\infty \in \mathbb{D}$ of the map $F_{\mathbb{D}}$ established in Lemma 4. Then, for $i = 0, \infty$, one of the two segments of the divisor $\mathbb{D}$ joining the critical point α_i and the fixed point p_i, must be contained in the basin of attraction of p_i.*

Proposition 2. *Let $F \in \mathcal{A}$ and $F_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$ be the restriction of its blow-up to the divisor. We consider the unique point $\mathsf{q}_1 \in \mathbb{D}$, $\mathsf{q}_1 \neq \mathsf{p}_1$, such that $F_{\mathbb{D}}(\mathsf{q}_1) = \mathsf{p}_1$, established in Corollary 2. Then the segments $(\mathsf{q}_1, \mathsf{p}_1)$, $(\mathsf{p}_1, \mathsf{q}_1) = \mathbb{D} \setminus [\mathsf{q}_1, \mathsf{p}_1]$ of the divisor $\mathbb{D}$, are invariant sets under $F_{\mathbb{D}}$.*

Proof. We focus on the case where $F \in \mathcal{A}_{\pm}$, since the analysis for the other families is analogous. We use the location of the points indicated in Lemma 4 and Corollary 2 (see Fig. 1, $F \in \mathcal{A}_{\pm}$). To simplify the notation, we identify each point in the divisor $\mathbb{D}$ with its expression in the chart u.

To prove the result, it suffices to show that the mapping $f(u)$ sends the interval $(\mathsf{q}_1, \mathsf{p}_1)$ into itself, and that the image of $(-\infty, \mathsf{q}_1)$, $(\mathsf{p}_1, \infty) \setminus \{\mathsf{q}_\infty\}$ under the mapping $f(u)$ is contained in the union $(-\infty, \mathsf{q}_1) \cup (\mathsf{p}_1, \infty)$. By Corollary 3 the interval $[\alpha_0, \mathsf{p}_0]$ is contained in the basin of attraction of p_0 with respect to $f(u)$, while the interval $[\alpha_\infty, \infty)$ is contained in the basin of attraction of $\mathsf{p}_\infty = \pm\infty$. The latter means that for all $a \in [\alpha_\infty, \infty)$, some point of its orbit, say $f^{n_0}(a)$, is equal to q_∞, or, alternatively, the absolute values of the points in its orbit $(|f^n(a)|)$ tends to infinity.

First, we prove that $f(\mathsf{q}_1, \mathsf{p}_1)$ is contained in $(\mathsf{q}_1, \mathsf{p}_1)$. For this purpose, we use the inequalities $\mathsf{q}_1 < \alpha_0 < \mathsf{p}_1$. On one hand, $f(\mathsf{q}_1, \alpha_0] = [f(\alpha_0), \mathsf{p}_1)$, since $f(u)$ is strictly decreasing in the interval $(-\infty, \alpha_0]$; on the other hand, $f[\alpha_0, \mathsf{p}_1) = [f(\alpha_0), \mathsf{p}_1)$, since $f(u)$ is strictly increasing in the interval $[\alpha_0, \mathsf{q}_\infty)$ (Lemma 4). It remains to prove that $\mathsf{q}_1 < f(\alpha_0)$. Since the interval $[\alpha_0, \mathsf{p}_0]$ is contained in the basin of attraction of p_0, then its image $f[\alpha_0, \mathsf{p}_0] = [f(\alpha_0), \mathsf{p}_0]$ is also contained in this basin; if we suppose that $\mathsf{q}_1 \geqslant f(\alpha_0)$, then $\mathsf{q}_1 \in [f(\alpha_0), \mathsf{p}_0]$ would be in the basin of attraction of p_0, wich is a contradiction since $\mathsf{p}_1 = f(\mathsf{q}_1)$ is a fixed point. Therefore, $\mathsf{q}_1 < f(\alpha_0)$ and the interval $[f(\alpha_0), \mathsf{p}_1)$ is contained in $(\mathsf{q}_1, \mathsf{p}_1)$. This enables us to conclude that $f(\mathsf{q}_1, \mathsf{p}_1)$ is contained in $(\mathsf{q}_1, \mathsf{p}_1)$.

Finally, we prove that the image of $(-\infty, \mathsf{q}_1)$ and $(\mathsf{p}_1, \infty) \setminus \{\mathsf{q}_\infty\}$ under $f(u)$, is contained in the union $(-\infty, \mathsf{q}_1) \cup (\mathsf{p}_1, \infty)$. On one hand, $f(-\infty, \mathsf{q}_1) = (\mathsf{p}_1, \infty)$, since $f(u)$ is strictly decreasing in the interval $(-\infty, \alpha_0]$. Now, it remains to prove that the image of $(\mathsf{p}_1, \infty) \setminus \{\mathsf{q}_\infty\}$ under $f(u)$, is contained in the union $(-\infty, \mathsf{q}_1) \cup (\mathsf{p}_1, \infty)$. Since the set $(\mathsf{p}_1, \infty) \setminus \{\mathsf{q}_\infty\}$ is equal to the union of the three intervals $(\mathsf{p}_1, \mathsf{q}_\infty)$, $(\mathsf{q}_\infty, \alpha_\infty]$, $[\alpha_\infty, \infty)$, then we have that

$$f(\mathsf{p}_1, \mathsf{q}_\infty) = (\mathsf{p}_1, \infty)\,, \quad f(\mathsf{q}_\infty, \alpha_\infty] = (-\infty, f(\alpha_\infty)]\,, \quad f[\alpha_\infty, \infty) = (-\infty, f(\alpha_\infty)]\,,$$

where the first two equalities are due to the fact that $f(u)$ is strictly increasing in each of the intervals $[\alpha_0, \mathsf{q}_\infty)$ and $(\mathsf{q}_\infty, \alpha_\infty]$, and the third equality follows from the fact that $f(u)$ is strictly decreasing in $[\alpha_\infty, \infty)$ (Lemma 4). It remains to prove that$f(\alpha_\infty) < \mathsf{q}_1$ to conclude that the interval $(-\infty, f(\alpha_\infty)]$ is contained in $(-\infty, \mathsf{q}_1)$. Since the interval $[\alpha_\infty, \infty)$ is contained in the basin of attraction of $\pm\infty$, then its image $f[\alpha_\infty, \infty) = (-\infty, f(\alpha_\infty)]$ is also contained in this basin. If we suppose that $\mathsf{q}_1 \leqslant f(\alpha_\infty)$, then $\mathsf{q}_1 \in (-\infty, f(\alpha_\infty)]$ would be in the basin of attraction of $\pm\infty$, which is a contradiction since $\mathsf{p}_1 = f(\mathsf{q}_1)$ is a fixed point. Therefore, $f(\alpha_\infty) < \mathsf{q}_1$ and the interval $(-\infty, f(\alpha_\infty)]$ is contained in $(-\infty, \mathsf{q}_1)$. This completes the proof. □

4.3. Basins of Attraction of p_0, p_∞

In 4.3.1 of this subsection, we will complete the proof of Theorem 7. For this purpose, in addition to the properties given in the previous subsections, we use Lemma 6 and Lemma 7.

Lemma 6. *Let $a, b, c, \mathsf{m} \in \mathbb{R}$ such that $a < \mathsf{m} < c < b$. Let $\varphi\colon [a, b] \to [a, b]$ be a continuous map with the following properties:*

(i) *the map φ has only two fixed points, namely, the points b, c;*

(ii) *the map φ is strictly decreasing in the interval $[a, \mathsf{m}]$, and it is strictly increasing in the interval $[\mathsf{m}, b]$;*

(iii) *there exists one point $x_0 \in (c, b)$ whose orbit $\{\varphi^n(x_0)\}$ converges to the fixed point c.*

Then the interval (a, b) is contained in the basin of attraction of the fixed point c, that is, for each $x \in (a, b)$ its orbit $\{\varphi^n(x)\}$ converges to c.

Proof. First, we focus on the interval (c, b). Note that the map φ restricted to the interval $[c, b]$, $\varphi|_{[c,b]}\colon [c, b] \to [c, b]$, is a strictly increasing homeomorphism. Since $\varphi|_{[c,b]}$ has no other fixed points than c and b, there exist only two options: $\varphi(x) < x$ for every $x \in (c, b)$, or $\varphi(x) > x$ for every $x \in (c, b)$.

We know that there exists $x_0 \in (c, b)$ whose orbit $\{\varphi^n(x_0)\}$ converges to c. Then $\varphi(x) < x$, for every $x \in (c, b)$. Hence, for every $x \in (c, b)$, its orbit $\{\varphi^n(x)\}$ converges to c.

On the other hand, $x < \varphi(x)$ for every $x \in [a, c)$, since $a < \varphi(a)$ and φ has no fixed points in the interval $[a, c)$. As a consequence, $\mathsf{m} < \varphi(\mathsf{m})$. Moreover, for every x in the interval (m, c), where φ is strictly increasing, $\mathsf{m} < \varphi(\mathsf{m}) < \varphi(x) < c$ and $x < \varphi(x)$; therefore, for every $x \in [\mathsf{m}, c)$ its orbit converges to c.

So far we conclude that $[\mathsf{m}, b) = [\mathsf{m}, c] \cup (c, b)$ is contained in the basin of attraction of c. Now we consider the interval $(a, \mathsf{m}]$, where φ is strictly decreasing. Since $\varphi(a, \mathsf{m}] = [\varphi(\mathsf{m}), \varphi(a))$ and $\mathsf{m} < \varphi(\mathsf{m})$, then $\varphi(a, \mathsf{m}]$ is contained in (m, b); hence, $(a, \mathsf{m}]$ is contained in the basin of attraction of c. Thus we conclude that for every $x \in (a, b) = (a, \mathsf{m}] \cup [\mathsf{m}, b)$ its orbit $\{\varphi^n(x)\}$ converges to c, as required. □

Lemma 7. *Let $a, b, c, \mathsf{m} \in \mathbb{R}$ such that $a < c < \mathsf{m} < b$. Let $\varphi\colon [a, b] \to [a, b]$ be a continuous map with the following properties:*

(i) *the map φ has only two fixed points, namely, the points b, c;*

(ii) *the map φ is strictly decreasing in the interval $[a, \mathsf{m}]$, and it is strictly increasing in the interval $[\mathsf{m}, b]$;*

(iii) *the interval $[c, \mathsf{m}]$ is contained in the basin of attraction of the fixed point c, that is, for each $x \in [c, \mathsf{m}]$ its orbit $\{\varphi^n(x)\}$ converges to c.*

Then the interval (a, b) is contained in the basin of attraction of the fixed point c, that is, for each $x \in (a, b)$ its orbit $\{\varphi^n(x)\}$ converges to c.

Proof. Since the map φ is strictly decreasing in the interval $[a, \mathsf{m}]$, and $a < c < \mathsf{m}$, then $\varphi(\mathsf{m}) < \varphi(c) = c$. Whereas φ is strictly increasing in the interval $[\mathsf{m}, b]$, there exists $c_1 \in (\mathsf{m}, b)$ such that $\varphi(c_1) = c$.

Let $U(c)$ be the basin of attraction of c. We know that $[c, \mathsf{m}]$ is contained in $U(c)$; then so is its image $\varphi[c, \mathsf{m}] = [\varphi(\mathsf{m}), c]$. Since the image of $[\mathsf{m}, c_1]$ under φ is again the interval $[\varphi(\mathsf{m}), c]$, then $[\mathsf{m}, c_1]$ is also contained in $U(c)$. Therefore, $[\varphi(\mathsf{m}), c_1] = [\varphi(\mathsf{m}), c] \cup [c, \mathsf{m}] \cup [\mathsf{m}, c_1]$ is contained in $U(c)$.

Now we consider the interval $[c, b)$. Since $c = \varphi(c_1) < c_1 < b$, and φ is strictly increasing in the interval $[c_1, b]$, then there exists c_2 in (c_1, b) satisfying $\varphi(c_2) = c_1$. Since $\varphi([c_1, c_2]) = [c, c_1]$, thus $[c_1, c_2]$ is contained in $U(c)$. Hence, the entire interval $[c, c_2] = [c, c_1] \cup [c_1, c_2]$ is also contained in $U(c)$. We assume that there are $n+1$ points $c_1, c_2, \dots, c_n, c_{n+1}$, which together with $c_0 := c$, satisfy:

— $\mathsf{m} < c_i < c_{i+1} < b$, for $0 \leqslant i \leqslant n$;

— $\varphi(c_{i+1}) = c_i$ and $\varphi([c_i, c_{i+1}]) = [c_{i-1}, c_i]$, for $1 \leqslant i \leqslant n$;

— the interval $[c, c_{n+1}]$ is contained in $U(c)$.

Since $\varphi(c_{n+1}) < c_{n+1} < b$, and φ is strictly increasing in the interval $[c_{n+1}, b]$, then there exists c_{n+2} in (c_{n+1}, b) such that $\varphi(c_{n+2}) = c_{n+1}$. Since the image of the interval $[c_{n+1}, c_{n+2}]$ under φ is equal to $[c_n, c_{n+1}]$, then the interval $[c_{n+1}, c_{n+2}]$ is contained in $U(c)$. Therefore, the entire interval $[c, c_{n+2}] = [c, c_{n+1}] \cup [c_{n+1}, c_{n+2}]$ is also contained in $U(c)$.

Let $\{c_n\}$ be the sequence of points described above. Since it is a strictly increasing sequence, then it converges to its supremum β. Then $c < \beta \leqslant b$, and moreover

$$\varphi(\beta) = \varphi\left(\lim_{n\to\infty} c_n\right) = \lim_{n\to\infty} \varphi(c_n) = \lim_{n\to\infty} c_{n-1} = \beta\,.$$

Since β is a fixed point of φ in $(c, b]$, then $\beta = b$. Therefore, $[c, b)$ is contained in $U(c)$, the basin of attraction of c.

So far we have that $[\varphi(\mathsf{m}), b) = [\varphi(\mathsf{m}), c] \cup [c, b)$ is contained in $U(c)$. Since φ is strictly decreasing in $(a, \mathsf{m}]$, then $\varphi((a, \mathsf{m}]) = [\varphi(\mathsf{m}), \varphi(a))$, which is contained in $[\varphi(\mathsf{m}), b)$. Therefore, $(a, \mathsf{m}]$ is contained in $U(c)$. Since $\varphi(\mathsf{m}) < c < \mathsf{m}$, then $(a, b) = (a, \mathsf{m}] \cup [\varphi(\mathsf{m}), b)$ is contained in $U(c)$, as required. □

4.3.1. Conclusion of Theorem 7

Let F be a map in the family $\mathcal{A}$, and $F_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ the restriction of its blow-up to the divisor. To complete the proof of Theorem 7 we need to show the following properties:

— there exists a unique point $\mathsf{q}_1 \in \mathbb{D}$, $\mathsf{q}_1 \neq \mathsf{p}_1$, such that $F_{\mathbb{D}}(\mathsf{q}_1) = \mathsf{p}_1$;

— the punctured divisor $\mathbb{D} \setminus \{\mathsf{p}_1, \mathsf{q}_1\}$ consists of two segments, which are invariant sets under $F_{\mathbb{D}}$;

— each attractor of $F_{\mathbb{D}}$ lies in one of the segments of $\mathbb{D} \setminus \{\mathsf{p}_1, \mathsf{q}_1\}$, which turns out to be its basin of attraction.

Corollary 2 enables us to obtain the location of the unique point $\mathsf{q}_1 \in \mathbb{D}$, $\mathsf{q}_1 \neq \mathsf{p}_1$, such that $F_{\mathbb{D}}(\mathsf{q}_1) = \mathsf{p}_1$. Furthermore, Proposition 2 claims that the segments $(\mathsf{q}_1, \mathsf{p}_1)$, $(\mathsf{p}_1, \mathsf{q}_1)$ are invariants subsets of the divisor under the map $F_{\mathbb{D}}$.

Let us recall the mappings $f(u)$ and $\hat{f}(v)$ given in (1), which represent $F_{\mathbb{D}}$ from the chart u to itself, and from the chart v to itself, respectively. The following properties are direct consequences of Lemma 4 and Corollary 3.

— Let $F \in \mathcal{A}_+$. The restriction of $f(u)$ to the segment $[\mathsf{q}_1, \mathsf{p}_1]$ seen from the chart u, and the restriction of $\hat{f}(v)$ to the segment $[\mathsf{p}_1, \mathsf{q}_1]$ seen from the chart v, satisfy the conditions of Lemma 6. Therefore, the basin of attraction of p_0 is the segment $(\mathsf{q}_1, \mathsf{p}_1)$, and the basin of attraction of p_∞ is the segment $(\mathsf{p}_1, \mathsf{q}_1)$.

— Let $F \in \mathcal{A}_-$. The restriction of $f(u)$ to the segment $[\mathsf{q}_1, \mathsf{p}_1]$ seen from the chart u, and the restriction of $\hat{f}(v)$ to the segment $[\mathsf{p}_1, \mathsf{q}_1]$ seen from the chart v, satisfy the conditions of Lemma 7. Therefore, the basin of attraction of p_0 is the segment $(\mathsf{q}_1, \mathsf{p}_1)$, and the basin of attraction of p_∞ is the segment $(\mathsf{p}_1, \mathsf{q}_1)$.

— Let $F \in \mathcal{A}_{\pm}$. The restriction of $f(u)$ to the segment $[\mathsf{q}_1, \mathsf{p}_1]$ seen from the chart u satisfies the conditions of Lemma 6, while the restriction of $\hat{f}(v)$ to the segment $[\mathsf{p}_1, \mathsf{q}_1]$ seen from the chart v, satisfies the conditions of Lemma 7. Therefore, the basin of attraction of p_0 is the segment $(\mathsf{q}_1, \mathsf{p}_1)$, and the basin of attraction of p_∞ is the segment $(\mathsf{p}_1, \mathsf{q}_1)$.

This completes the proof of Theorem 7.

References

1. **Katok A., Hasselblatt B.** Introduction to the Modern Theory of Dynamical Systems. *Encyclopedia of Mathematics and its Applications*. Vol. 54. Cambridge, Cambridge University Press, 1995.
2. **Conway J.B.** *Functions of One Complex Variable*. I. New York, Springer-Verlag, 1978.
3. **Devaney R.** *Introduction to Chaotic Dynamical Systems.* Menlo Park, Benjamin Cummings, 1986.
4. **Block L.S., Coppel W.A.** *Dynamics in One Dimension.* Berlin, Heidelberg, Springer-Verlag, 1992.
5. **Brin M., Stuck G.** *Introduction to Dynamical Systems.* Cambridge, Cambridge University Press, 2002.
6. **Robinson C.** *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos.* Boca Raton, CRC Press, 1995.
7. **Milnor J.** *Dynamics in One Complex Variable.* New Jersey, Princeton University Press, 2006.

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ДИНАМИКА СЕМЕЙСТВА ОТОБРАЖЕНИЙ, ОПРЕДЕЛЯЕМЫХ КВАДРАТИЧНЫМИ ПОЛИНОМАМИ

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Мы рассматриваем отображения $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, координаты которых являются однородными многочленами в $\mathbb{R}[x, y]$ степени 2. Эти карты отображают линии, проходящие через начало координат, в линии, проходящие через начало координат. Наша цель — изучить, как эти линии перемещаются под действием F. Мы покажем, что существует действительное аналитическое многообразие $\mathcal{F}^2$, где можно чётко различить два множества. Одно множество $\mathcal{U} \subseteq \mathcal{F}^2$ состоит из преобразований, которые имеют «скрытую гиперболическую» динамику, а его дополнение $\mathcal{F}^2 \setminus \mathcal{U}$ содержит карты, которые показывают хаотичное поведение.

Ключевые слова: *полиномиальное отображение, круговое отображение, хаотическая динамика.*

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