Chelyabinsk Physical and Mathematical Journal. 2017. Vol. 2, iss. 2. P. 210–230.

УДК 517.55+517.53

BOUNDED INTEGRAL OPERATORS IN SOME WEIGHTED SPACES OF ANALYTIC FUNCTIONS ON PRODUCT DOMAINS AND RELATED RESULTS

R. F. Shamoyan[a], **S. P. Maksakov**[b]

Bryansk State University, Bryansk, Russia
[a]*rshamoyan@mail.ru;* [b]*msp222@mail.ru*

The expository article overviews new research results presented by the authors and their colleagues regarding to the existence problem of bounded projections of functions on weighted spaces in n-dimensional complex space C^n. New extensions are discussed. Some new interesting problems in analytic spaces on product domains are also provided. In addition, we shortly review new results of other authors working in this direction.

Keywords: *bounded projection, weighted space, integral representation, polydisk, unit ball, polyball, pseudoconvex domain, tubular domain.*

Introduction

Theorems on bounded linear projections in various weighted spaces from measurable functions into the corresponding spaces of analytic or harmonic functions play an important role in the study of various issues related to the space of analytic functions. The development of a number of very important issues in the theory of analytic functions of one and several variables (boundary properties, the problem of approximation and interpolation, the description of dual spaces, etc.) depends heavily on theorems on integral representations and the existence of bounded projectors in such type spaces (see [1], for example).

This paper considers new issues in general spaces. Classes of analytic functions f in the disk $U = \{z \in C : |z| < 1\}$ such that

$$A^p_\alpha(U) = \left\{ f : \|f\|_{A^p_\alpha(U)} = \left(\int\limits_U |f(z)|^p \, (1 - |z|)^\alpha \, dm_2(z) \right)^{1/p} < +\infty \right\},$$

where $0 < p < +\infty$, dm_2 is the Lebesgue measure in U, $\alpha > -1$ were strictly defined and studied in [2; 3] for the first time in literature.

Note, however, for $p = 2$ case this was studied much earlier by Bergman, and Hardy and Littlewood provided various one side estimates for some norms (quasinorms) of more general form. Though strict definitions in these classes can not be found in their papers.

In [4; 5] F. A. Shamoyan introduced new Bergman type anisotropic weighted spaces of analytic functions in a polydisc $U^n = \{z = (z_1, \ldots, z_n) : |z_j| < 1,\ j = 1, 2, \ldots, n\}$ and studied the question of integral representations, the existence of bounded projectors, duality in these spaces. These studies were continued in the works of F.A. Shamoyan and his disciples.

The main purpose of this article is to list results obtained by us and our colleagues in the past two decades related to bounded integral operators in weighted spaces of analytic functions in domains of multidimensional complex space $\mathbb{C}^n$ in the polydisc, unit ball, polyball, tubular domain and pseudoconvex domains.

The special attention in this paper is given to some special weights (see [6]) and related analytic function spaces: the study of various weighted analytic spaces is a known problem (see [1]).

Also, this paper may be served as a model for further research of analytic spaces in product domains, where unit disk is replaced by more complicated domains of tubular type, or unit ball, or pseudoconvex domain, in various expressions below.

Obviously, a necessity of writing such a review arose since we and our colleagues conducted rather intensive research in this area. Many results are presented in Russian non peer-reviewed journals or in young scientists' theses.

Our technique and methods may be considered as its far reaching development of the early works [4; 5] of F.A. Shamoyan, when much simpler cases were considered.

Some results of this paper can be probably generalized to analytic spaces by analogous methods, when the unit disk is replaced by more general domain. Although, these issues are not discussed in detail in this paper.

In conclusion, note that this work may attract the attention of various experts who actively work in the area of complex and functional analysis, when their professional interests intersect with the mentioned problems.

In this paper, firstly we consider bounded integral operators in weighted spaces on the polydisc. Then they are considered on the unit ball and the polyball and finally, in the tubular and pseudoconvex domains.

The final part of the paper is denoted to a short review of some recent results.

There are also mentioned the interesting applications to extremal problems and trace theorem, obtained by the first author.

Note finally that the list of publications, dedicated to this problem, is quite long. In this paper we focus on our research interest areas.

1. Bounded integral operators in weighted spaces in polydisc

In this section, we study the above questions in weighted spaces of analytic and n-harmonic, pluriharmonic functions in the polydisc, simplest product domains. This section may serve a model for further research, since the unit polydisc is the simplest product domain.

Let $U^n = \{z \in \mathbb{C}^n, z = (z_1, \dots, z_n) : |z_j| < 1, j = 1, 2, \dots, n\}$ be a unit polydisc in n-dimensional complex space C^n; $U = U^1$;

$$T^n = \{z \in \mathbb{C}^n, z = (z_1, \dots, z_n) : |z_j| = 1, j = 1, 2, \dots, n\} -$$

be the Shilov boundary of U^n; $T = T^1$; $H(U^n)$ be a set of all analytic functions in U^n; $H^p(U^n)$ be the Hardy class in U^n, $h^p(U^n)$ be the Hardy class of a harmonic function in U^n.

Let us denote a set of all functions $\omega(t) = (\omega_1(t), \dots, \omega_n(t))$ by Ω^n (see [6]), $\omega_j(t)$, $j = 1, 2, \dots, n$, are positive integrable on the interval $(0, 1)$ functions, so that there exist positive numbers m_ω, M_ω, q_ω, and $q_\omega \in (0, 1)$ such that

$$m_\omega \leq \frac{\omega_j(\lambda t)}{\omega_j(t)} \leq M_\omega \quad \forall t \in (0, 1),\ \lambda \in [q_\omega, 1],\ j = 1, 2, \dots, n.$$

Suppose that $\omega_j \in \Omega$. Let $\alpha_\omega = \dfrac{\ln m_\omega}{\ln \frac{1}{q_\omega}}$, $\beta_\omega = \dfrac{\ln M_\omega}{\ln \frac{1}{q_\omega}}$; $\alpha_\omega > -1$, $0 < \beta_\omega < 1$.

If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, then $z^\alpha := z^{\alpha_1} \cdot \dots \cdot z^{\alpha_n}$, $|\alpha| := \alpha_1 + \dots + \alpha_n$. In addition, put

$$\left(1-|z|^2\right)^\alpha := \prod_{j=1}^n \left(1-|z_j|^2\right)^{\alpha_j}; \quad (1-\zeta z)^\alpha := \prod_{j=1}^n (1-\zeta_j z_j)^{\alpha_j};$$

$$\omega_\Pi(1-|z|) := \prod_{j=1}^n \omega_j(1-|z_j|); \quad \omega_\Pi^s(1-|z|) := \prod_{j=1}^n \omega_\Pi^s(1-|z_j|), \, s \in \mathbb{R}.$$

Let also for $\alpha_j > -1$, $\omega_j \in \Omega$, $0 < p_j < +\infty$, $j = 1, 2, \dots, n$

$$\omega_{\alpha_j}(t) := \omega_j(t) \left(\frac{t^{\alpha_j}}{\omega_j(t)}\right)^{p_j}, \quad \omega_\alpha(t) = (\omega_{\alpha_1}(t), \dots, \omega_{\alpha_n}(t)), \quad t \in (0,1),$$

$$\frac{\Gamma(\alpha+1+k)}{\Gamma(\alpha+1)\Gamma(k+1)} = \prod_{j=1}^n \frac{\Gamma(\alpha_j+1+k_j)}{\Gamma(\alpha_j+1)\Gamma(k_j+1)},$$

where Γ is the well-known Euler function.

Let also

$$D_\alpha(z, \bar{\zeta}) = \prod_{j=1}^n \frac{\alpha_j+1}{\pi} \frac{(1-|\zeta_j|^2)^{\alpha_j}}{(1-z_j\bar{\zeta}_j)^{\alpha_j+2}};$$

$$K_\alpha(z, \bar{\zeta}) = \prod_{j=1}^n \left(\frac{1}{(1-z_j\bar{\zeta}_j)^{\alpha_j+2}} + \frac{1}{(1-\bar{z}_j\zeta_j)^{\alpha_j+2}} - 1 \right);$$

$$E_\alpha(z, \bar{\zeta}) = \frac{1}{(1-z\bar{\zeta})^{\alpha+2}} + \frac{1}{(1-\bar{z}\zeta)^{\alpha+2}} - 1$$

for $\alpha_j > -1$, $j = 1, 2, \dots, n$.

Denote by $c, c_1, \dots, c_n(\alpha, \beta, \dots), \dots$ arbitrary positive constants, depending on $\alpha, \beta, \dots$, where specific values play no role.

We need the following classes of function.

Let $L^{\vec{p}}_\omega(U^n)$, $\omega \in \Omega^n$, $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $j = 1, 2, \dots, n$, be a set of measurable functions f in U^n and

$$\|f\|_{L^{\vec{p}}_\omega(U^n)} = \left(\int_U \omega_n(1-|z_n|) \left(\int_U \omega_{n-1}(1-|z_{n-1}|) \dots \right.\right.$$

$$\left.\left. \dots \left(\int_U |f(z_1, \dots, z_n)|^{p_1} \omega_1(1-|z_1|) dm_2(z_1) \right)^{p_2/p_1} \dots \right)^{p_n/p_{n-1}} dm_2(z_n) \right)^{1/p_n} < +\infty.$$

We denote a subspace of $L^{\vec{p}}_\omega(U^n)$, consisting of analytic functions, by $A^{\vec{p}}_\omega(U^n)$; a subspace of $L^{\vec{p}}_\omega(U^n)$, consisting of n-harmonic functions, by $h^{\vec{p}}_\omega(U^n)$; a subspace of $L^{\vec{p}}_\omega(U^n)$, consisting of pluriharmonic functions, by $\tilde{h}^{\vec{p}}_\omega(U^n)$. These are Banach spaces when $\min\limits_{1\le j\le n} p_j \ge 1$, otherwise they are complete metric spaces (see [7]).

Recall a function $u(z_1,\dots,z_n)$ as n-harmonic in U^n, if it is harmonic on each variable, i.e. $u(z_1,\dots,z_n)$ is a solution of equations

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} = 0, \quad j = 1,2,\dots,n.$$

A function $u(z_1,\dots,z_n)$ is pluriharmonic in U^n, if it is a real part of an analytic function

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \quad j,k = 1,2,\dots,n.$$

Let also $L^{\vec{p}}(U^n) = L^{\vec{p}}_\omega(U^n)$, $A^{\vec{p}}(U^n) = A^{\vec{p}}_\omega(U^n)$, $h^{\vec{p}}(U^n) = h^{\vec{p}}_\omega(U^n)$, $\tilde{h}^{\vec{p}}(U^n) = \tilde{h}^{\vec{p}}_\omega(U^n)$ if $\omega_j(t) \equiv 1$, $t \in (0,1)$, $\omega_j \in \Omega$, $j = 1,2,\dots,n$.

New absorbing results on weighted spaces projections with mixed norm of analytic, n-harmonic, pluriharmonic functions in the unit polydisc are formulated in [7; 8] by F.A. Shamoyan, O.V. Yaroslavtseva.

Summing up the results with respect to spaces of analytic functions, let us formulate the following result.

Theorem 1. *Let $\omega = (\omega_1,\dots,\omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1,\dots,p_n)$, $1 < p_j < +\infty$, $\alpha_j > \alpha_{\omega_j}$, $j = 1,2,\dots,n$. Then an operator*

$$G_{\alpha,\omega}(f)(z) = \int\limits_{U^n} \frac{f(\zeta)\omega(1-|\zeta|)}{\left(1-\overline{\zeta}z\right)^{\alpha+2}} dm_{2n}(\zeta), \quad z \in U^n, \tag{1}$$

maps the space $L^{\vec{p}}_\omega(U^n)$ into $A^{\vec{p}}_{\omega_\alpha}(U^n)$, and an estimate

$$\|G_{\alpha,\omega}(f)\|_{A^{\vec{p}}_{\omega_\alpha}(U^n)} \le c\,\|f\|_{L^{\vec{p}}_\omega(U^n)} \tag{2}$$

is valid.

Here and further dm_{2n} is the Lebesgue measure in U^n. As a consequence of this theorem the following assertion is obtained.

Corollary 1. *Let $\omega = (\omega_1,\dots,\omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1,\dots,p_n)$, $1 < p_j < +\infty$, $\alpha_j > \alpha_{\omega_j}$, $j = 1,2,\dots,n$. Then the operator*

$$G_\alpha(f)(z) = \int\limits_{U^n} \frac{f(\zeta)(1-|\zeta|^2)^\alpha}{\left(1-\overline{\zeta}z\right)^{\alpha+2}} dm_{2n}(\zeta), \quad z \in U^n, \tag{3}$$

maps the space $L^{\vec{p}}_\omega(U^n)$ into $A^{\vec{p}}_\omega(U^n)$ as a bounded operator.

From this theorem it is obtained:

Corollary 2. *Let $\omega = (\omega_1,\dots,\omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1,\dots,p_n)$, $1 < p_j < +\infty$, $\alpha_j > -1$, $j = 1,2,\dots,n$. Then the operator* (3) *maps the space $L^{\vec{p}}(U^n)$ into $A^{\vec{p}}(U^n)$.*

When $0 < p_j \le 1$, $j = 1,2,\dots,n$ the following theorem holds:

Theorem 2. *Let $\omega = (\omega_1,\dots,\omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1,\dots,p_n)$, $0 < p_j \le 1$, $\alpha_j > \dfrac{\alpha_{\omega_j}+2}{p_j} - 1$, $j = 1,2,\dots,n$. Then operator*

$$G_\alpha(f)(z) = \int\limits_{U^n} f(\zeta) D_\alpha(\zeta,z) dm_{2n}(\zeta), \quad z \in U^n, \tag{4}$$

is bounded from $h^{\vec{p}}_{\omega}(U^n)$ *into* $A^{\vec{p}}_{\omega_\alpha}(U^n)$.

The following important estimate is obtained from theorems 1 and 2.

Corollary 3. *Let* $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $\alpha_j > \dfrac{\alpha_{\omega_j} + 2}{p_j} - 1$, $j = 1, 2, \dots, n$, $f \in H(U^n)$ *and* $\mathrm{Im} f(0) = 0$. *Then*

$$\|f\|_{A^{\vec{p}}_{\omega}(U^n)} \le c \, \|\mathrm{Re} f\|_{h^{\vec{p}}_{\omega}(U^n)} .$$

These estimates are well-known in L^p-norm, namely in spaces of analytic functions defined in a ball for $1 < p < +\infty$ and in a polydisc for $0 < p < +\infty$ (see, for example, [5; 9; 10]).

For spaces of n-harmonic functions and pluriharmonic functions the following assertions are valid.

Theorem 3. *Let* $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, \dots, p_n)$, $1 < p_j < +\infty$, $\alpha_j > \alpha_{\omega_j}$, $j = 1, 2, \dots, n$. *Then*

1) *the operator*

$$M_{\alpha,\omega}(f)(z) = \int\limits_{U^n} f(\zeta)\omega(1 - |\zeta|) K_\alpha(z, \bar{\zeta}) dm_{2n}(\zeta), \quad z \in U^n,$$

maps the space $L^{\vec{p}}_{\omega}(U^n)$ *into* $h^{\vec{p}}_{\omega_\alpha}(U^n)$, *and* $\|M_{\alpha,\omega}(f)\|_{h^{\vec{p}}_{\omega_\alpha}(U^n)} \le c \, \|f\|_{L^{\vec{p}}_{\omega}(U^n)}$;

2) *the operator*

$$N_{\alpha,\omega}(f)(z) = \int\limits_{U^n} f(\zeta)\omega(1 - |\zeta|) E_\alpha(z, \bar{\zeta}) dm_{2n}(\zeta), \quad z \in U^n,$$

maps the space $L^{\vec{p}}_{\omega}(U^n)$ *into* $\tilde{h}^{\vec{p}}_{\omega_\alpha}(U^n)$, *and* $\|N_{\alpha,\omega}(f)\|_{\tilde{h}^{\vec{p}}_{\omega_\alpha}(U^n)} \le c \, \|f\|_{L^{\vec{p}}_{\omega}(U^n)}$.

In particular, the following theorem is valid.

Theorem 4. *Let* $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, \dots, p_n)$, $1 < p_j < +\infty$, $\alpha_j > \alpha_{\omega_j}$, $j = 1, 2, \dots, n$. *Then*

1) *the operator*

$$M_\alpha(f)(z) = \int\limits_{U^n} f(\zeta)(1 - |\zeta|)^\alpha K_\alpha(z, \bar{\zeta}) dm_{2n}(\zeta), \quad z \in U^n, \tag{5}$$

maps the space $L^{\vec{p}}_{\omega}(U^n)$ *into* $h^{\vec{p}}_{\omega}(U^n)$;

2) *the operator*

$$N_\alpha(f)(z) = \int\limits_{U^n} f(\zeta)(1 - |\zeta|)^\alpha E_\alpha(z, \bar{\zeta}) dm_{2n}(\zeta), \quad z \in U^n, \tag{6}$$

maps the space $L^{\vec{p}}_{\omega}(U^n)$ *into* $\tilde{h}^{\vec{p}}_{\omega}(U^n)$.

From the previous theorem we immediately get:

Corollary 4. *Let* $\vec{p} = (p_1, \dots, p_n)$, $1 < p_j < +\infty$, $\alpha_j > -1$, $j = 1, 2, \dots, n$. *Then*

1) *operator* (5) *maps the space* $L^{\vec{p}}(U^n)$ *into* $h^{\vec{p}}(U^n)$;

2) *operator* (6) *maps the space* $L^{\vec{p}}(U^n)$ *into* $\tilde{h}^{\vec{p}}(U^n)$.

Anisotropic weighted spaces of analytic functions in a polydisc with mixed norm are studied in [11; 12] by F. A. Shamoyan, N. A. Chasova, refining several results of [7; 8]. A bounded linear operator mapping n-harmonic functions space $h^{\vec{p}}_{\omega}(U^n)$ on $A^{\vec{p}}_{\omega}(U^n)$ for all $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $j = 1, 2, \dots, n$ is constructed. The main result is presented by the following theorem.

Theorem 5. *Let* $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $\alpha_j > \dfrac{\alpha_{\omega_j} + 2}{p_j} - 1$, $j = 1, 2, \dots, n$. *Then operator* (4) *maps the space* $h^{\vec{p}}_{\omega}(U^n)$ *into* $A^{\vec{p}}_{\omega}(U^n)$, *and* $\|G_\alpha(f)\|_{A^{\vec{p}}_{\omega}(U^n)} \le c\,\|f\|_{h^{\vec{p}}_{\omega}(U^n)}$.

Further, we denote by $L^{\vec{p}}(T^n)$, $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $j = 1, 2, \dots, n$, a set of measurable functions f in U^n with a finite quasinorm

$$\|f\|_{L^{\vec{p}}(T^n)} = \left(\int\limits_T \dots \left(\int\limits_T \left(\int\limits_T |f(\vec{z})|^{p_1}\, d\sigma(z_1) \right)^{p_2/p_1} \dots \right)^{p_3/p_2} \dots d\sigma(z_n) \right)^{1/p_n},$$

where $d\sigma$ is the Lebesque measure on T.

Let $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $j = 1, 2, \dots, n$. Then we denote generalized Hardy spaces with mixed norm $H^{\vec{p}}(U^n)$ as a set of analytic functions f in U^n, for which

$$\sup_{0 \le r < 1} \left(\int\limits_T \dots \left(\int\limits_T \left(\int\limits_T |f(r\vec{z})|^{p_1}\, d\sigma(z_1) \right)^{p_2/p_1} \dots \right)^{p_3/p_2} \dots d\sigma(z_n) \right)^{1/p_n} < +\infty.$$

When $p_1 = \dots = p_n = p$, the space $H^{\vec{p}}(U^n)$ coincides with the usual Hardy space $H^p(U^n)$ (see [1]). These are Banach spaces for $\min\limits_{1 \le j \le n} p_j \ge 1$.

An amusing and vital problem is studying the action of Bergman projection in these Hardy-type spaces when $\min\limits_{1 \le j \le n} p_j \ge 1$. The same problem obviously can be raised in the polyball and even in more general tubular and pseudoconvex domains. We omit definitions of parallel spaces in these domains leaving that to readers. We refer to [13–15] for some results in this direction.

We define also the Poisson kernel as the product

$$P(z, \zeta) = P_{r_1}(\theta_1 - \varphi_1) \cdots P_{r_n}(\theta_n - \varphi_n),$$

where $\zeta = (\zeta_1, \dots, \zeta_n) \in T^n$, $\zeta_j = e^{i\varphi_j}$; $j = 1, 2, \dots, n$; $z = (z_1, \dots, z_n) \in U^n$, $z_j = r_j e^{i\varphi_j}$, $j = 1, 2, \dots, n$;

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

is a standard Poisson kernel for U.

In [11; 12] the authors studied the generalized Hardy space in the polydisc with mixed norm. They give a complete characterization of some spaces of n-harmonic functions via products of one-dimensional Poisson integrals and kernels, and construct a bounded projector, which maps the space $L^{\vec{p}}(T^n)$ onto $H^{\vec{p}}(U^n)$ for $\vec{p} = (p_1, \dots, p_n)$, $1 < p_j < +\infty$, $j = 1, 2, \dots, n$.

Theorem 6. *Let* $\vec{p} = (p_1, \dots, p_n)$, $1 < p_j < +\infty$, $j = 1, 2, \dots, n$. *Let also* $\zeta = (\zeta_1, \dots, \zeta_n) \in T^n$, $\zeta_j = e^{i\varphi_j}$; $j = 1, 2, \dots, n$; $z = (z_1, \dots, z_n) \in U^n$, $z_j = r_j e^{i\varphi_j}$, $j = 1, 2, \dots, n$. *Then the operator*

$$I(f)(r_1 e^{i\varphi_1}, \dots, r_n e^{i\varphi_n}) = \frac{1}{(2\pi i)^n} \int\limits_{T^n} \frac{f(\theta_1, \dots, \theta_n) d\theta_1 \dots d\theta_n}{(e^{i\theta_1} - r_1 e^{i\varphi_1}) \dots (e^{i\theta_n} - r_n e^{i\varphi_n})}$$

maps the space $L^{\vec{p}}(T^n)$ *into* $H^{\vec{p}}(U^n)$, *and* $\|I(f)\|_{H^{\vec{p}}(U^n)} \le C \|f\|_{L^{\vec{p}}(T^n)}$.

In [16] O.E. Antonenkova studied $L^{\vec{p},\vec{q}}_{\omega}(U^n)$ spaces, $\omega \in \Omega$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n)$, $1 \le p_j, q_j < +\infty$, $j = 1, 2, \dots, n$. It is the set of measurable functions f in U^n, for which

$$\|f\|_{L^{\vec{p},\vec{q}}_{\omega}(U^n)} = \left(\int\limits_0^1 \omega_n(1 - r_n) \left(\int\limits_{-\pi}^{\pi} \dots\dots \left(\int\limits_0^1 \omega_1(1 - r_1) \times \right.\right.\right.$$

$$\left.\left.\left. \times \left(\int\limits_{-\pi}^{\pi} \left| f(r_1 e^{i\varphi_1}, \dots, r_n e^{i\varphi_n}) \right|^{p_1} d\varphi_1 \right)^{q_1/p_1} dr_1 \right)^{p_2/q_1} \dots d\varphi_n \right)^{q_n/p_n} dr_n \right)^{1/q_n} < +\infty.$$

These are Banach function spaces (see [16]).

We denote as usual the subspace $L^{\vec{p},\vec{q}}_{\omega}(U^n)$ consisting of analytic functions by $A^{\vec{p},\vec{q}}_{\omega}(U^n)$. In case $p_j = q_j$ these spaces were introduced in [7]. In [16] the author constructed a bounded projection from space $L^{\vec{p},\vec{q}}_{\omega}(U^n)$ to $A^{\vec{p},\vec{q}}_{\omega_\alpha}(U^n)$ for all $1 \le p_j, q_j < +\infty$, and $\omega_\alpha(t) = (\omega_{\alpha_1}(t), \dots, \omega_{\alpha_n}(t))$, $\omega_{\alpha_j}(t) := \omega_j(t)(t^{\alpha_j/\omega_j(t)})^{q_j}$, $t \in (0, 1)$, $\alpha_j > -1$, $\omega_j \in \Omega$, $j = 1, 2, \dots, n$.

Theorem 7. *Let* $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n)$, $1 \le p_j, q_j < +\infty$, $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \Omega$, $\omega_{\alpha_j}(t) := \omega_j(t)(t^{\alpha_j}/\omega_j(t))^{q_j}$, $t \in (0, 1)$, $\alpha_j > \alpha_{\omega_j}$, $j = 1, 2, \dots, n$. *Then operator* (1) *maps the space* $L^{\vec{p},\vec{q}}_{\omega}(U^n)$ *into* $A^{\vec{p},\vec{q}}_{\omega_\alpha}(U^n)$, *and the estimate*

$$\|G_{\alpha,\omega}(f)\|_{A^{\vec{p},\vec{q}}_{\omega_\alpha}(U^n)} \le c(\vec{p}, \vec{q}, \alpha) \|f\|_{L^{\vec{p},\vec{q}}_{\omega}(U^n)}$$

is valid.

In recent paper [17] E.V. Povprits considers the $\tilde{L}^{p,q}_{\omega}(U^n)$ space, $\omega \in \Omega^n$, $0 < p, q < +\infty$, as the set of measurable functions f in U^n, for which

$$\|f\|_{\tilde{L}^{p,q}_{\omega}} = \left(\int\limits_{Q_n} \omega\,(1 - r) \left(\int\limits_{T^n} |f(rz)|^p \, d\sigma_n(z) \right)^{q/p} dr \right)^{1/q} < +\infty,$$

where $d\sigma_n$ is the Lebesgue measure in T^n, $Q^n = (0, 1]^n$. We also introduce the notation $\tilde{A}^{p,q}_{\omega}(U^n) = H(U^n) \bigcap \tilde{L}^{p,q}_{\omega}(U^n)$ with the corresponding quasi-norm. These are Banach spaces for $\min\limits_{1 \le j \le n} (p, q) \ge 1$ and complete metric spaces for other values of parameters (see [17]).

Many projection theorems were proved in [17], we provide a typical result.

Theorem 8. *Let* $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$, $1 < p_j, q_j < +\infty$, $\omega = (\omega_1, \ldots, \omega_n)$, $\omega_j \in \Omega, \omega_{\alpha_j}(t) := \omega_j(t)(t^{\alpha_j}/\omega_j(t))^{q_j}$, $j = 1, 2, \ldots, n$, $t \in (0, 1)$, $\alpha > \alpha_\omega$. *Then the operator*

$$G_{\alpha,\omega_\Pi}(f)(z) = \int\limits_{U^n} \frac{f(\zeta)\omega_\Pi(1-|\zeta|)}{\left(1-\overline{\zeta}z\right)^{\alpha+2}} dm_{2n}(\zeta), \quad z \in U^n,$$

maps the space $\tilde{L}^{\vec{p},\vec{q}}_{\omega}(U^n)$ *into* $\tilde{A}^{\vec{p},\vec{q}}_{\omega_\alpha}(U^n)$, *and the estimate*

$$\|G_{\alpha,\omega_\Pi}(f)\|_{\tilde{A}^{\vec{p},\vec{q}}_{\omega_\alpha}(U^n)} \le c(\vec{p}, \vec{q}, \alpha) \, \|f\|_{\tilde{L}^{\vec{p},\vec{q}}_{\omega}(U^n)}$$

is valid.

Remark 1. We shall introduce in the second section some new spaces of functions in products of unit disks (polydisc). The study of direct analogues of same mixed norm Hardy, Bergman type spaces, but in products of tubular domains over symmetric cones and bounded strictly pseudoconvex domains is an open and very interesting problem. Moreover, the same mixed norm Hardy spaces in harmonic function classes and pluriharmonic classes deserves also a separate study in more complicated domains. Only some results provided in this and next sections.

Consider expanded Bergman projection in the polydisc

$$(EP)(f)(\vec{z}) = \int\limits_{U} \frac{f(\zeta)(1-|\zeta|)^\alpha dm_2(\zeta)}{\prod\limits_{j=1}^{n}(1-\zeta\overline{z_j})^{\alpha+2}}, \quad \vec{z} = (z_1, \ldots, z_n), \ z_j \in U, \ j = 1, 2, \ldots, n.$$

These operators and their direct extensions to various domains play very important role in trace theorems (see [18–21]).

Consider Bergman type projections on special type (on semiproducts of domains)

$$(SP)(f)(\vec{z}) = \int\limits_{T^n} \int\limits_{0}^{1} \frac{f(\zeta)(1-|\zeta|)^\alpha d\tilde{m}_2(\zeta)}{\prod\limits_{j=1}^{n}(1-z_j\,|\zeta|\,\varphi_j)^{\alpha+2}}, \quad \vec{z} = (z_1, \ldots, z_n), \ z_j \in U, \ j = 1, 2, \ldots, n,$$

or similar when integration moves by $T \times [0;1]^n$. Various results and applications on these type integral operators in various domains can be seen in [22; 23].

2. Bounded integral operators in weighted spaces in the unit ball and the unit polyball

In this section we give some new assertions on boundedness of integral operators in weight spaces of analytic functions with mixed norm in the unit ball and polyball (product of unit balls).

Let $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be a unit ball in n-dimension complex plane $\mathbb{C}^n$, S_n be the boundary of B_n. Let also $L^{p,q}_\omega(B_n)$, $\omega \in \Omega$, $0 < p, q < +\infty$, be the set of measurable functions f in B_n, for which

$$\|f\|_{L^{p,q}_\omega(B_n)} = \left(\int\limits_0^1 \omega\,(1-r) \left(\int\limits_{S_n} |f(rz)|^p \, d\sigma(z) \right)^{q/p} r^{2n-1} dr \right)^{1/q} < +\infty,$$

where $d\sigma$ this is the Lebesque measure in S_n.

We denote by $H(B_n)$ the class of analytic functions f in B_n; $A^{p,q}_\omega(B_n) = H(B_n)\bigcap L^{p,q}_\omega(B_n)$. Denote also the subspace of $L^{p,q}_\omega(B_n)$ consisting of n-harmonic functions by $h^{p,q}_\omega(B_n)$.

In [16; 24] O.E. Antonenkova, F.A. Shamoyan constructed a bounded projection from the space $L^{p,q}_\omega(B_n)$ to the space $A^{p,q}_\omega(B_n)$ for all $1 \le p,q < +\infty$ and from the space $h^{p,q}_\omega(B_n)$ to the space $A^{p,q}_\omega(B_n)$ for all $0 < p,q < +\infty$. The following theorem holds:

Theorem 9. *Let $1 \le p,q < +\infty$, $\omega \in \Omega$, $\alpha > \alpha_\omega$. Then the operator*

$$Q_\alpha(f)(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int\limits_{B_n} \frac{f(\zeta)(1-|\zeta|^2)^\alpha}{(1-\langle\zeta,z\rangle)^{\alpha+n+1}} d\nu(\zeta), \quad z \in B_n, \tag{7}$$

maps the space $L^{p,q}_\omega(B_n)$ into $A^{p,q}_\omega(B_n)$, and the estimate $\|Q_\alpha(f)\|_{A^{p,q}_\omega(B_n)} \le c\,\|f\|_{L^{p,q}_\omega(B_n)}$ is valid.

Note, here and below, $d\nu$ is the Lebesque measure in B_n, Γ is the Euler function.

Remark 2. In the special case, where $1 < p,q < +\infty$, $\omega(t) = t^\alpha$, $\alpha > -1$, this result was obtained in [25]. Direct complete analogues of these results in unbounded domains namely in tube domains is an open problem. Only partial results are known (see [26] and references there).

In the case of the other kernel the following assertion is valid:

Theorem 10. *Let $1 \le p,q < +\infty$, $\omega \in \Omega$, $\omega_\alpha(t) := \omega(t)(t^\alpha/\omega(t))^q$, $t \in (0,1)$, $\alpha > \alpha_\omega$. Then the operator*

$$Q_{\alpha,\omega}(f)(z) = c(\alpha,w) \int\limits_{B_n} \frac{f(\zeta)\omega(1-|\zeta|)}{(1-\langle\zeta,z\rangle)^{\alpha+n+1}} d\nu(\zeta), \quad z \in B_n, \tag{8}$$

maps the space $L^{p,q}_\omega(B_n)$ into $A^{p,q}_{\omega_\alpha}(B_n)$, and the estimate $\|Q_{\alpha,\omega}(f)\|_{A^{p,q}_{\omega_\alpha}(B_n)} \le c\,\|f\|_{L^{p,q}_\omega(B_n)}$ is valid.

For $h^{p,q}_\omega(B_n)$ spaces the same authors received the following three results:

Theorem 11. *Let $1 \le p,q < +\infty$, $\omega \in \Omega$,*

$$\alpha > \frac{\alpha_\omega + 1}{\min\{p,q\}} + n\left(\frac{1}{p} - 1\right) - 1.$$

Then operator (7) maps the space $h^{p,q}_\omega(B_n)$ into $A^{p,q}_\omega(B_n)$, and the estimate

$$\|Q_\alpha(f)\|_{A^{p,q}_\omega(B_n)} \le c\,\|f\|_{h^{p,q}_\omega(B_n)} \tag{9}$$

is valid.

Theorem 12. *Let $0 \le p,q < 1$, $\omega \in \Omega$, $\alpha > (\alpha_\omega + 1)/q - 1$. Then operator (7) maps the space $h^{p,q}_\omega(B^n)$ into $A^{p,q}_\omega(B^n)$, and estimate (9) is valid.*

Corollary 5. *Let $0 \le p,q < 1$, $\omega \in \Omega$,*

$$\alpha > \frac{\alpha_\omega + 1}{q} + n\left(\frac{1}{p} - 1\right) - 1.$$

If

$$\int\limits_0^1 \omega^p(1-r)(1-r)^{(n+1)(p-1)} dr < +\infty,$$

then operator (8) *maps the space* $h^{p,q}_{\omega}(B_n)$ *into* $A^{p,q}_{\omega_\alpha}(B_n)$, *and the estimate*

$$\|Q_{\alpha,\omega}(f)\|_{A^{p,q}_{\omega_\alpha}(B_n)} \le c\,\|f\|_{h^{p,q}_{\omega}(B_n)} \tag{10}$$

is valid.

Theorem 13. *Let* $\omega \in \Omega$. *Suppose that*

1) *if* $1 < p < +\infty$, $0 < q \le 1$, *then* $\alpha > \dfrac{\alpha_\omega + 1}{q} - 1$;

2) *if* $0 < p \le 1$, $1 < q < +\infty$, *then* $\alpha > \dfrac{\alpha_\omega + 1}{p} + n\left(\dfrac{1}{p} - 1\right) - 1$.

Then operator (7) *maps the space* $h^{p,q}_{\omega}(B_n)$ *into* $A^{p,q}_{\omega}(B_n)$, *and estimate* (9) *is valid.*

Corollary 6. *Let* $\omega \in \Omega$. *Suppose that*

1) *if* $1 < p < +\infty$, $0 < q \le 1$, *then* $\alpha > \dfrac{\alpha_\omega + 1}{q} - 1$;

2) *if* $0 < p \le 1$, $1 < q < +\infty$, *then* $\alpha > \dfrac{\alpha_\omega + 1}{p} + n\left(\dfrac{1}{p} - 1\right) - 1$.

Then operator (8) *maps the space* $h^{p,q}_{\omega}(B_n)$ *into* $A^{p,q}_{\omega_\alpha}(B^n)$, *and estimate* (10) *is valid.*

Remark 3. Various weighted analytic spaces in the unit ball were studied in recent decades. We refer the reader for various other projector theorems for similar weighted mixed norm spaces in the unit ball and in the unit disk to the [27], where other types of weights were considered in mixed norm spaces.

Let B_n^m denote the polyball $B_n^m = B_n \times B_n \times \cdots \times B_n$. Let also $S_n^m = S_n \times S_n \times \cdots \times S_n$. As usual, we denote by $H(B_n^m)$ the space of all analytic functions in B_n^m by each variable separately. Introduce the mixed norm classes in polyballs

$$A^{\vec{p}}_{\vec{\alpha}}(B_n^m) = \left\{ f \in H(B_n^m) : \|f\|_{A^{\vec{p}}_{\vec{\alpha}}} := \left(\int_{B_n} (1-|z_m|)^{\alpha_m} \left(\int_{B_n} (1-|z_{m-1}|)^{\alpha_{m-1}} \dots \right.\right.\right.$$

$$\left.\left.\left. \dots \left(\int_{B_n} |f(z_1,\dots,z_m)|^{p_1} (1-|z_1|)^{\alpha_1} d\nu(z_1) \right)^{p_2/p_1} \dots d\nu(z_{m-1}) \right)^{p_m/m-1} d\nu(z_m) \right)^{1/p_m} < +\infty \right\},$$

where $0 < p_j < \infty$, $\alpha_j > -1$, $j = 1, 2, \dots, n$.

Note that for $n = 1$ these classes studied in [7]. For $m = 1$ we have the classical Bergman spaces on the unit ball. Formally replacing B_n by R^n we arrive at well studied function classes in R^n (see [28; 29]).

It is not difficult to show that $A^{\vec{p}}_{\vec{\alpha}}$ is a Banach space for $1 \le p_j < \infty$, $j = 1, 2, \dots, m$. Moreover, it can be shown that $A^{\vec{p}}_{\vec{\alpha}}$ is a complete metric space if all $0 < p_j < 1$, $j = 1, 2, \dots, m$.

In [30] the authors proved projection and trace and duality theorems on these spaces. For $n = 1$ or $m = 1$ (polydisk case) these results are known (see [7; 31]).

We define $d\nu_m$ the Lebesque measure in B_n^m, as $d\nu_m(\zeta) = d\nu(\zeta_1)\dots d\nu(\zeta_m)$, $\zeta = (\zeta_1,\dots,\zeta_m)$, $\zeta_j \in B_n$, $j = 1, 2, \dots, m$.

Theorem 14. *Let $\vec{p}=(p_1,\dots,p_m)$, $1<p_j<+\infty$, $\alpha=(\alpha_1,\dots,\alpha_m)$, $\alpha_j>-1$, $j=1,2,\dots,m$. Then the operator*

$$R_{\vec{\alpha}}(f)(z)=\int\limits_{B_n^m}\frac{f(\zeta)\prod\limits_{j=1}^{m}(1-|\zeta_j|)^{\alpha_j}}{\prod\limits_{j=1}^{m}(1-z_j\bar{\zeta}_j)^{\alpha_j+n+1}}d\nu_m(\zeta),\quad z\in B_n^m,$$

maps the space $L^{\vec{p}}_{\vec{\alpha}}(B_n^m)$ into $A^{\vec{p}}_{\vec{\alpha}}(B_n^m)$, and the estimate $\|R_{\vec{\alpha}}(f)\|_{A^{\vec{p}}_{\vec{\alpha}}(B_n^m)}\le c\,\|f\|_{L^{\vec{p}}_{\vec{\alpha}}(B_n^m)}$ is valid.

3. Bounded integral operators in weighted Bergman type spaces in the tubular and pseudoconvex domain

In this section we add some new theorems on the projections for mixed norm spaces in tubular domains and pseudoconvex domains obtained by the first author and his coauthors. For formulation of our results we will need a series of notations and definitions which are standard for the case of analytic spaces in tubular domains over symmetric cones and strongly pseudoconvex domains with smooth boundary in C^n.

Let $T_\Omega=V+i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space $\widetilde{V}$. We denote the rank of the cone Ω by r and by Δ the determinant function on $\widetilde{V}$. Letting $\widetilde{V}=\mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the forward light cone Λ_n defined for $n\ge 3$ by $\Lambda_n=\{y\in\mathbb{R}^n: y_1^2-\cdots-y_n^2>0, y_1>0\}$. Light cones have rank 2. The determinant function in this case is given by the Lorentz form $\Delta(y)=y_1^2-\cdots-y_n^2$.

$H(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. We denote m Cartesian products of tubes by T_Ω^m: $T_\Omega^m=T_\Omega\times\cdots\times T_\Omega$. The space of all analytic functions on this new product domain which are analytic by each variable separately will be denoted by $H(T_\Omega^m)$. In this paper we will be interested on properties of certain analytic subspaces of $H(T_\Omega^m)$. By m we denote below a natural number, $m>1$.

For $\tau\in\mathbb{R}_+$ and the associated determinant function $\Delta(x)$ we set

$$A_\tau^\infty(T_\Omega)=\left\{F\in H(T_\Omega):\|F\|_{A_\tau^\infty}=\sup_{x+iy\in T_\Omega}|F(x+iy)|\Delta^\tau(y)<+\infty\right\}.$$

It can be checked that this is a Banach space.

For $1\le p,q<+\infty$ and $\nu\in\mathbb{R}$, $\nu>n/r-1$, we denote by $A_\nu^{p,q}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of analytic functions f in T_Ω that

$$\|F\|_{A_\nu^{p,q}(T_\Omega)}=\left(\int\limits_{\Omega}\left(\int\limits_{\widetilde{V}}|F(x+iy)|^p dx\right)^{q/p}\frac{\Delta^\nu(y)}{\Delta(y)^{n/r}}dy\right)^{1/q}<+\infty.$$

We put $A_\nu^{p,p}(T_\Omega):=A_\nu^p(T_\Omega)$, $1\le p\le+\infty$. This is a Banach space. Replacing above simply A by L we will get as usual the corresponding larger space of all measurable functions in tube over symmetric cone with the same quasinorm (see, for example, [26]).

To define related two Bergman-type spaces $A_\nu^p(T_\Omega^m)$ and $A_\tau^\infty(T_\Omega^m)$ (ν and τ can be also vectors) in m-products of tube domains T_Ω^m we follow standard procedure which is well-known in the case of the unit disk and the unit ball (see, e.g., [18]). Namely we consider analytic F functions $F=F(z_1,\dots z_m)$ which are analytic by each z_j, $j=1,2,\dots,m$, variable, and where each such variable belongs to T_Ω tube. For example we set, for all $z_j = x_j + y_j$, $F(z) = F(z_1, \dots, z_m)$, $\tau = (\tau_1, \dots, \tau_m)$, $\tau_j \in \mathbb{R}$, $j = 1, 2, \dots, m$,

$$A_\tau^\infty(T_\Omega^m) = \left\{ F \in H(T_\Omega^m) : \|F\|_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega^m} |F(x+iy)| \Delta^\tau(y) < \infty \right\},$$

$$|F(x+iy)| = |F(x_1 + iy_1, \dots, x_m + iy_m)|,$$

where $\Delta^\tau(y)$ is a product of m one-dimensional $\Delta^{\tau_j}(y_j)$ functions, $j = 1, 2, \dots, m$.

Similarly the Bergman space A_τ^p can be defined on products of tubes T_Ω^m for all $\tau = (\tau_1, \dots, \tau_m)$, $\tau_j > n/r - 1$, $j = 1, 2, \dots, m$. It can be shown that all these spaces are Banach spaces. Replacing A by L we will get as usual the corresponding larger space of all measurable functions in products of tubes over symmetric cone with the same quasinorm.

Further we will use the standard notation (see, e.g., [32])

$$dV_\nu(w) = \Delta^{\nu - n/r}(v) du dv, \quad w = u + iv \in T_\Omega, \quad z = x + iy \in T_\Omega.$$

We add some basic definitions for pseudoconvex domains.

Let k_Δ denote the Poincare distance on the unit disk $U \subset \mathbb{C}^n$. If X is a complex manifold, the Lempert function δ_X: $X \times X \to \mathbb{R}^+$ of X is defined by

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) | \exists \phi \in H(U),\, \phi : U \to X; \phi(\zeta) = z, \phi(\eta) = w\},$$

for all $z, w \in X$.

We denote by D a bounded strictly pseudoconvex domain with a smooth boundary $\delta(z) = \operatorname{dist}(z, \partial D)$, $dv(z)$ is a Lebeques measure on D, dv_β is a weighted measure $dv_\beta = \delta^\beta dv$.

Let also $D^m = D \times \dots \times D$, $H(D^m)$ is a class of all analytic functions on D^m.

Let $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of D, the K_t is a weighted kernel of type t. Note $K = K_{n+1}$.

We define new Banach mixed norm Bergman type spaces in products of pseudoconvex domains and mixed norm Bergman-type spaces in products of tubular domains over symmetric cones as follows.

Let $m \geq 1$, $p_j \in (1, \infty)$, $\nu_j > n/r - 1$, $\alpha_j > -1$, $j = 1, 2, \dots, m$,

$$A_{\vec{\nu}}^{\vec{p}}(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \left(\int_{T_\Omega} \dots \left(\int_{T_\Omega} |f(z_1, \dots, z_m)|^{p_1} \left[\Delta^{\nu_1 - n/r}(y_1)\right] dx_1 dy_1 \right)^{p_2/p_1} \dots \right.\right.$$

$$\left.\left. \dots \left[\Delta^{\nu_m - n/r}(y_m)\right] dx_m dy_m \right)^{1/p_m} < +\infty \right\},$$

$$A_{\vec{\alpha}}^{\vec{p}}(D^m) = \left\{ f \in H(D^m) : \left(\int_D \dots \left(\int_D |f(z_1, \dots, z_m)|^{p_1} \delta^{\alpha_1}(z_1) dv(z_1) \right)^{p_2/p_1} \dots \right.\right.$$

$$\left.\left. \dots \delta^{\alpha_m}(z_m) dv(z_m) \right)^{1/p_m} < +\infty \right\}.$$

Replacing A by L we get larger spaces of measurable functions with the same norms.

The following type theorem has a direct application to the trace problems in the tube domains (see [33]).

Theorem 15. *Let* $\overrightarrow{z} = (z_1, \dots, z_m)$, $z_j \in T_\Omega$, $\beta_j > \beta_0$, $j = 1, 2, \dots, m$, *for some fixed large enough* β_0. *Then*

1) *the operator*

$$T_{\overrightarrow{\beta}} f(\overrightarrow{z}) = \int\limits_{T_\Omega^m} \frac{f(w_1, \dots, w_m) dV_{\beta_1}(w_1) \dots dV_{\beta_m}(w_m)}{\Delta^{\beta_1 + \frac{n}{r}}(\frac{z_1 - \overline{w}_1}{i}) \dots \Delta^{\beta_m + \frac{n}{r}}(\frac{z_m - \overline{w}_m}{i})}$$

maps $L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(T^m_\Omega)$ *into* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(T^m_\Omega)$, $p_j > 1$, $\nu_j > n/r - 1$, $j = 1, 2, \dots, m$;

2) *the operator*

$$\widetilde{T}_\beta f(\overrightarrow{z}) = \int\limits_{T_\Omega} \frac{f(w) dV_\beta(w)}{\prod\limits_{j=1}^{m} \Delta^{\frac{\beta + \frac{n}{r}}{m}}(\frac{z_j - w}{i})}$$

maps $A^{p_m}_{\nu}(T_\Omega)$ *into* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(T^m_\Omega)$, $\nu = \sum\limits_{j=1}^{m} \left[\nu_j - \frac{n}{r} + \frac{2n}{r}(m-1) \right] \frac{p_m}{p_j}$.

For the unit ball or the unit disk case this theorem can be seen in [7; 8; 19]. For $p_j = p$, $j = 1, 2, \dots, m$, part 2) can be seen in [33].

We provide a complete analogue of the above theorem for bounded strictly pseudo-convex domains with smooth boundary.

Theorem 16. *Let* $\overrightarrow{z} = (z_1, \dots, z_m)$, $z_j \in D$, $\beta_j > \beta_0$, $j = 1, 2, \dots, m$, *for some fixed sufficiently large* β_0. *Then*

1) *the operator*

$$S_{\overrightarrow{\beta}} f(\overrightarrow{z}) = \int\limits_{D^m} f(w_1, \dots, w_m) K_{\beta_1}(z_1, w_1) \dots K_{\beta_m}(z_m, w_m) dv_{\beta_1}(w_1) \dots dv_{\beta_m}(w_m)$$

maps $L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(D^m)$ *into* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(D^m)$, $p_j > 1$, $\nu_j > -1$, $j = 1, 2, \dots, m$;

2) *the operator*

$$\widetilde{S}_\beta f(\overrightarrow{z}) = \int\limits_{D} f(w) \prod_{j=1}^{m} K_\beta(z_j, w) dv_\beta(w)$$

maps $A^{p_m}_{\nu}(D)$ *into* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(D^m)$, $\nu = \sum\limits_{j=1}^{m} \nu_j + (m-1)(n-1)$.

For $p_j = p$, $j = 1, 2, \dots, m$, this theorem (the second part) can be seen in [18; 19; 34] in the unit disk and ball. For the unit disk case theorems (the first parts) can be seen in [7; 8]. Also the proof in all cases is almost parallel to the unit disk case for $p_j = p$, $j = 1, 2, \dots, m$ (see [34]).

The main goal of [22; 33] is to continue investigation on Bergman type multifunctional spaces, related to analytic spaces on product of tube domains and Bergman type integral operators on them. The authors define new Bergman type spaces in product domains and apply Whitney type decomposition of cone and tubular domain in the studying of Bergman type operators on them.

We formulate first some new results on Bergman type integral operators now in Bergman type spaces, related results in Herz type spaces are also valid.

Theorem 17. *Let*

$$[(R_{x,y} g)(w)] = \left[(\operatorname{Im} w)^{-m\frac{2n}{r} + \sum\limits_{j=1}^{m} y_j} \right] \times$$

$$\times \int\limits_{T_\Omega} \dots \int\limits_{T_\Omega} g(z_1, \dots, z_m) \frac{\left[\prod_{j=1}^{m} (\mathrm{Im} z_j)^{x_j}\right]}{\prod_{j=1}^{m} \left|\Delta^{r_j+y_j}\left(\frac{\omega - z_j}{i}\right)\right|} dv(z_1)\dots dv(z_m)$$

for $g \in L^1(T_\Omega^m, dv(z_1)\dots dv(z_m))$, $\omega \in T_\Omega$. *Let* $1 \le p < +\infty$, $s_j > n/r - 1$, $s_j + 1 < p(x_j + 1)$, $ms_j + 1 > mp(\frac{2n}{r} - y_j) - \frac{2n}{r}(m-1)$, *for each* $j = 1, 2, \dots, m$.

Then there is a constant $c > 0$ *such that*

$$\int\limits_{T_\Omega} |(R_{x,y}g)(\omega)|^p (\mathrm{Im}\omega)^{(m-1)\frac{2n}{r} + \sum_{j=1}^{m}(s_j - \frac{n}{r})} dv(\omega) \le$$

$$\le c \int\limits_{T_\Omega} \dots \int\limits_{T_\Omega} |g(z_1, \dots, z_m)|^p \prod_{j=1}^{m} (\mathrm{Im} z)^{s_j - \frac{n}{r}} dv(z_j).$$

Remark 4. Theorem 16 in the case of the unit ball of $\mathbb{C}^n$ were proved in [18].
Remark 5. The reverse operator has a form

$$(T_\beta f)(\vec{z}) = \int\limits_{T_\Omega} f(z) \left[\frac{\Delta^{\frac{1}{m}\sum_{j=1}^{m}\beta_j - \frac{n}{r}}(\mathrm{Im} z)}{\Delta^{\frac{1}{m}\sum_{j=1}^{m}\beta_j + \frac{n}{r}}\left(\frac{z - z_j}{i}\right)}\right] dv(z),$$

$z_j \in T_\Omega$, $\beta_j > n/r - 1$, $j = 1, 2, \dots, m$, and its properties were considered recently in [33].

Next, let $1 \le p < +\infty$, $f = f(z_1, \dots, z_m)$, we consider the following analytic spaces for $f \in H(T_\Omega^m)$, $\nu \ge n/r - 1$, $\nu_j \ge n/r - 1$, $j = 1, 2, \dots, m$. These are spaces with norms

$$\|f\|^p_{(A^p_{\vec{\nu}})_1} = \int\limits_{T_\Omega} \dots \int\limits_{T_\Omega} |f(x_1 + iy_1, \dots, x_m + iy_m)|^p \prod_{j=1}^{m} \Delta^{\nu_j - \frac{n}{r}}(y_j)\, dx_j\, dy_j < +\infty,$$

$$\|f\|^p_{(A^p_\nu)_2} = \int\limits_{R^n} \dots \int\limits_{R^n} \int\limits_{\Omega} |f(x_1 + iy, \dots, x_m + iy)|^p \Delta^{\nu - \frac{n}{r}}(y) \prod_{j=1}^{m} dx_j \dots dy < +\infty,$$

$$\|f\|^p_{(A^p_{\vec{\nu}})_3} = \int\limits_{R^n} \int\limits_{\Omega} \dots \int\limits_{\Omega} |f(x + iy_1, \dots, x + iy_m)|^p \prod_{j=1}^{m} \Delta^{\nu_j - \frac{n}{r}}(y_j)\, dx\, dy_j < +\infty.$$

Let $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_j > 0$, $j = \overline{1, m}$; $\overrightarrow{\beta} = (\beta_1, \dots, \beta_m)$, $\beta_j > n/r - 1$, or $\overrightarrow{\beta} = (\beta, \dots, \beta)$, $\beta > n/r - 1$. Let also for $g \in L^1(T_\Omega^m)$

$$(V_{\vec{\alpha},\vec{\beta}}\, g)(\vec{w}) = \underbrace{\int\limits_{\Omega} \dots \int\limits_{\Omega}}_{m} \int\limits_{R^n} \frac{g(x + iy_1, \dots, x + iy_m)(\Delta y_1)^{\beta_1} \dots (\Delta y_m)^{\beta_m} dx}{\prod_{j=1}^{m} \Delta^{\alpha_j}\left(\frac{\bar{x} + i\bar{y}_j - w_j}{i}\right)} dy_1 \dots dy_m,$$

$$(U_{\vec{\alpha},\beta}\, g)(\vec{w}) = \underbrace{\int\limits_{R^n} \dots \int\limits_{R^n}}_{m} \int\limits_{\Omega} \frac{g(x_1 + iy, \dots, x_m + iy)(\Delta y)^{\beta}}{\prod_{j=1}^{m} \Delta^{\alpha_j}\left(\frac{\bar{x} + i\bar{y}_j - w_j}{i}\right)}\, dy\, dx_1 \dots dx_m,$$

$\vec{w} = (w_1, \dots, w_m)$, $\omega_j \in T_\Omega$, $j = 1, 2, \dots, m$. Define also

$$[G_{\vec{\alpha},\beta}(g)](x + iy_1, \dots, x + iy_m) = \int\limits_{T_\Omega} \frac{g(w)\Delta^\beta(\mathrm{Im} w)dv(w)}{\prod\limits_{j=1}^{m} \Delta^{\alpha_j}(\frac{w-(\bar{x}+\bar{y}_j)}{i})},$$

$x \in R^n$, $y_j \in \Omega$, $j = 1, 2, \dots, m$;

$$[\tilde{G}_{\vec{\alpha},\beta}(g)](x_1 + iy, \dots, x_m + iy) = \int\limits_{T_\Omega} \frac{g(w)\Delta^\beta(\mathrm{Im} w)dv(w)}{\prod\limits_{j=1}^{m} \Delta^{\alpha_j}(\frac{w-(\bar{x}_j+i\bar{y})}{i})},$$

$x_j \in R^n$, $j = 1, 2, \dots, m$, $y \in \Omega$. We give a typical result from [22].

Theorem 18. *For $1 \le p < \infty$, some $\alpha_j \in (\alpha_0, \alpha'_0)$; $\beta_j \in (\beta_0, \beta'_0)$, $\beta \in (\widetilde{\beta}_0, \widetilde{\beta'}_0)$, $\nu_j > n/r - 1$, $\tau_j > n/r - 1$, $j = 1, 2, \dots, m$, for some fixed positive α_0^j, $(\alpha_0^j)'$, β_0^j, $(\beta_0^j)'$.*

The following estimates are valid:

1) $\|G_{\vec{\alpha},\beta}(g)\|_{(A^p_{\vec{\nu}})_3} \le c_1 \|g\|_{(A^p_\tau)(T_\Omega)}$;
2) $\|\tilde{G}_{\vec{\alpha},\beta}(g)\|_{(A^p_\nu)_2} \le c_2 \|g\|_{(A^p_\tau)(T_\Omega)}$;
3) $\|V_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_{\vec{\nu}})_1} \le c_3 \|g\|_{(A^p_{\vec{\tau}})_3}$;
4) $\|U_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_{\vec{\nu}})_1} \le c_4 \|g\|_{(A^p_\tau)_2}$,

where $\alpha_0^j, \dots, \tilde{\beta}_0^j$ depend on p, n, ν_j, τ_j, $j = 1, 2, \dots, m$.

Remark 6. The analogue of Theorem 17 for unit disk can be seen in [23].

Remark 7. For less general case where R^n is I, $I = (0, 1)$ is the unit interval and $\Omega = T = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle such type results were proved in [18; 23].

Similar to Theorem 18 results are probably valid in pseudoconvex domains.

4. Bergman type projections in Hardy-type, BMOA-type and Bloch-type spaces and related problems

The natural and vital question is to study Bergman type projections in other spaces of analytic functions (Hardy-type, BMOA-type, Bloch-type). These type problems also have many applications (see [13–15; 35–38]), for example to the trace and the distance problems (see [19]).

Some results were obtained recently in this direction by the first author in tube and pseudoconvex domains (see [19]) for BMOA-type classes.

In simpler case of the unit disk similar results can be seen in [35; 36] and in [38].

For Bloch spaces we refer to [31; 34–36].

In a series of papers of W. Cohn, J. Ortega, J. Fabrega Bergman projection was studied in some Hardy-type spaces and $F^{p,q}_\alpha$ type spaces ($F^{p,2}_0 = H^p$) in the unit ball and even in pseudoconvex domains. A nice problem is to extend these results to the case of product domains in $F^{p,q}_\alpha$ type spaces in $\underbrace{C^n \times \dots \times C^n}_{m}$ and tube domains (see [13–15] and references there).

Bergman type projections are also studied in various Herz-type spaces in [20–22] in the polyball and other domains. Let us formulate some assertions in this direction related to the papers of the first author on BMOA-type spaces in the polyball.

Let $m \geq 1$. Consider an integral operator for any m-types of real numbers $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_m)$

$$(W_{a,b}f)(z_1, \dots, z_m) = \prod_{j=1}^{m}(1 - |z_j|^2)^{a_j} \int_{B_n} \frac{f(\zeta)(1 - |\zeta|^2)^{-n-1-\sum_{j=1}^{m} b_j}}{\prod_{j=1}^{m}(1 - \langle z_j, \zeta \rangle)^{a_j + b_j}} d\nu(\zeta),$$

where $z_j \in B_n$, $j = 1, 2, \dots, m$, $f \in L^1\left(B_n, d\nu_{-n-1-\sum_{j=1}^{m} b_j}\right)$ (expanded Bergman projection in the unit ball).

The following results show that this operator acts between two BMOA-type spaces in the unit ball and in the polyball. These results have applications to the trace problem.

Theorem 19. [19]. *Let $0 < p < +\infty$. Suppose $s_j > -1$, $r_j \geq 0$, $a_j > a_0$, $a_0 = a_0(s_1, \dots, s_m, r_1, \dots, r_m, n)$, $b_j > b_0$, $b_0 = b_0(s_1, \dots, s_m, r_1, \dots, r_m, n)$, $j = 1, 2, \dots, m$. Let also $t = (m-1)(n-1) + \sum_{j=1}^{m} s_j$. Then there is a constant c so that*

$$\int_{B_n} \dots \int_{B_n} |(W_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{s_j} d\nu(z_1) \dots d\nu(z_m)}{|1 - \langle u_j, z_j \rangle|^{r_j}} \leq$$

$$\leq c \int_{B_n} \frac{|f(w)|^p (1 - |w|^2)^t}{\left|\prod_{j=1}^{m}(1 - \langle u_j, w \rangle)^{r_j}\right|} d\nu(w).$$

These results are typical in some sense, since even more general results are valid in bounded symmetric domains, in tubular domains over symmetric cones and also in bounded strictly pseudoconvex domains with a smooth boundary in C^n. Under some condition on Bergman kernel, this is valid also in other domains.

5. Final remarks

In this section, several new general results of studying the action of Bergman projection are reviewed shortly. The spaces are concerned in C^n.

Bergman type projections that are studied in bounded symmetric domains are presented in [39] (see also references there); that are studied in Bergman classes are presented in a generalized unit disk in [40]; that are studied in Bergman classes are considered in Siegel domains in [41] (see also references here); that are studied in Bergman classes in homogeneous domains are presented in [42]; that are studied in Bergman classes with Mukenhoupt weights may be found in [27]; that are studied in Bloch type spaces in circular domains are considered in [43] (see also references there); that are studied in Besov-type spaces in admissible domains may be found, for example, in [44].

There is an open problem to extend all these results to mixed norm spaces on product domains. The methods of the previous section may be apparently used here.

A separate complex issue is the studying of Bergman type projection in Bergman spaces A^p_α in weakly pseudoconvex domains (pseudoconvex domains with various restrictions on Levi form of so-called defining function of a domain). The main proof tools here are so-called Forelli — Rudin type estimate and Schur test. We refer the reader to [45–47] (and references there) for recent results in this area. there are concerned bounded pseudoconvex domains which boundary points are of the finite type and with locally diagonalizable Levi form, or bounded pseudoconvex domains of the finite type with the property that Levi form of the boundary has at most one degenerate eigenvalue etc.

Finally, several useful applications on Bergman projections to weakly invertible functions in analytic spaces may be found in [48].

Bergman type projections that are studied in Besov spaces in bounded pseudoconvex domains with C^∞ boundary may be found in [49]. This problem is researched in [43] in Bloch-type spaces. The weighted Bergman projection in some Hartogs domains and weakly regular domains is studied in [50]. This issue that is studied in Reinhardt domains may be found in [51] (see also references there).

The problem of the extension of these type results to the case of parallel spaces, but on product domains is open.

Remark 8. For the one-dimensional case we mention [52; 53], where a projectors theorem is constructed in the weighted spaces of analytic and harmonic functions for the case of simply connected domains with angles, with an asymptotically conformal boundary and rectifiable boundary.

Note again, that these projections theorems may be used to construct new duality theorems to the case of more complicated domains such as tubular domains over symmetric cones and bounded strictly pseudoconvex domains with a smooth boundary and other domains.

We also mention papers [9; 54; 55] that contain projections theorems in Hardy and Bergman spaces in bounded symmetric domains and other domains. It will be nice to define their complete extensions as new mixed norm spaces on the product of bounded symmetric domains and to present projection results in these new spaces similarly as we have done in this paper in less general case of the unit disk or the unit ball.

Finally, we also note some recent interesting papers [56–62] on this topic.

References

1. **Shvedenko S.V.** Klassy Hardy i svyazannye s nimi prostranstva analiticheskikh funktsiy v yedinichnom kruge, polikruge i share [Hardy classes and related spaces of analytic functions in the unit disk, unit ball and polydisc]. *Itogi nauki i tekhniki. Seriya Matematicheskiy analiz* [Results of science and technics. Ser. Mathematical analysis], 1985, vol. 23, pp. 3–124. (In Russ.).
2. **Djrbashian M.M.** O kanonicheskom predstavlenii meromorfnykh funktsiy v yedinichnom diske [On canonical representation of meromorphic functions in the unit disk]. *Doklady Akademii nauk Armyanskoy SSR* [Reports of the Academy of sciences of Armenian SSR], 1945, vol. 3, no. 1, pp. 3–9. (In Russ.).
3. **Djrbashian M.M.** O probleme predstavleniya analiticheskikh funktsiy [On the problem of representation of analytic functions]. *Soobshsheniya Instituta matematiki i mekhaniki Akademii nauk Armyanskoy SSR* [Reports of Institute of Mathematics and Mechanics of Armenian SSR], 1948, vol. 2, pp. 3–35. (In Russ.).
4. **Shamoyan F.A.** Diagonal'nye otobrazheniya i nekotorye problemy predstavleniya v anizotropnykh prostranstvakh golomorfnykh v polidiske funktsiy [Diagonal mapping and some problems pf the representation in the anisotropic spaces of holomorphic in the polydisc functions]. *Doklady Akademii nauk Armyanskoy SSR* [Reports of the Academy of sciences of Armenian SSR], 1987, vol. 85, no. 1, pp. 21–26. (In Russ.).
5. **Shamoyan F.A.** Diagonal mapping and problems of representation in anisotropic spaces of holomorphic functions in the polydisk. *Siberian Mathematical Journal*, 1990, vol. 31, no. 2, pp. 350–365.

6. **Seneta E.** *Regularly Varying Functions*. Berlin, Heidelberg, New York, Springer-Verlag, 1976. 112 p.
7. **Shamoyan F.A., Yaroslavtseva O.V.** Nepreryvnye proyektsii, dvoystvennost' i diagonal'noye otobrazheniye v vesovykh prostranstvakh golomorfnykh funktsiy so smeshannoy normoy [Continuous projections, duality and the diagonal map in weighted spaces of holomorphic functions with mixed norm]. *Zapiski nauchnykh seminarov POMI* [Notes of the scientific workshops of POMI], 1997, vol. 247, pp. 268–275. (In Russ.).
8. **Yaroslavtseva O.V.** *Nekotorye voprosy predstavleniya v vesovykh prostranstvakh golomorfnykh i n-garmonicheskikh funktsiy so smeshannoy normoy* [Some issues of representation in the weighted spaces of holomorphic and the n-harmonic functions with the mixed norm. PhD Thesis]. Bryansk, 1999. (In Russ.).
9. **Coifman R., Rochberg R.** Representation theorems for holomorphic and harmonic functions in L_p. *Astérisque*, 1980, vol. 77, pp. 11–66.
10. **Forelli F., Rudin W.** Projections on spaces of holomorphic functions in balls. *Indiana University Mathematical Journal*, 1975, vol. 24, pp. 593–602.
11. **Shamoyan F.A., Chasova N.A.** Bounded projectors and continuous linear functionals on the weighted spaces of analytic functions with mixed norm in a polydisc. Preprint, Bryansk State University, Bryansk. 2002. (In Russ.).
12. **Chasova N.A.** *Integral'noye predstavleniye, dvoystvennost' i operatory Toeplitz'a v vesovykh anizotropnykh prostranstvakh golomorfnykh funktsiy so smeshannoy normoy v polidiske* [Integral representation, duality and Toeplitz operators in weighted anisotropic spaces of holomorphic functions with mixed norm in a polydisc. PhD Thesis]. Bryansk, 2003. (In Russ.).
13. **Cohn W.S.** Weighted Bergman projection and tangential area integrals. *Studia Mathematica*, 1993, vol. 106, no. 1, pp. 59–76.
14. **Cohn W.S.** Bergman projection and Hardy spaces. *Journal of Functional Analysis*, 1997, vol. 144, pp. 1–19.
15. **Ortega J., Fabrega J.** Holomorphic Triebel — Lizorkin type spaces. *Journal of Functional Analysis*, 1997, vol. 151, no. 1, pp. 177–212.
16. **Antonenkova O.E.** *Vosproizvodyashchiye yadra, preobrazovaniye Koshi nepreryvnykh lineynykh funktsionalov na vesovykh prostranstvakh golomorfnykh funktsiy* [Reproducing kernels, Cauchy transform of continuous linear functionals on the weighted spaces of holomorphic functions. PhD Thesis]. Bryansk, 2005. (In Russ.).
17. **Povprits E.V.** *Kharakterizatsiya sledov i preobrazovaniye Koshi nepreryvnykh lineynykh funktsionalov v anizotropnykh vesovykh prostranstvakh analiticheskikh funktsiy so smeshannoy normoy* [Characterization of the traces and the Cauchy transform continuous linear functionals in anisotropic weighted spaces of analytic functions with mixed norm. PhD Thesis]. Bryansk, 2015. (In Russ.).
18. **Shamoyan R.F., Mihic O.** On traces of holomorphic functions in polyballs. *Applied Analysis and Discrete Mathematics*, 2009, vol. 3, pp. 42–48.
19. **Shamoyan R.F., Mihic O.** On traces of Q_p type spaces and mixed norm analytic function spaces in polyballs. *Siauliau Mathematical Seminar*, 2010, vol. 5, no. 13, pp. 101–119.
20. **Shamoyan R.F., Mihic O.** In search of traces of some holomorphic spaces on polyballs. *Revista Notas Venezolana*, 2010, vol. 4, pp. 1–23.
21. **Shamoyan R.F., Kurilenko S.M.** On traces of analytic Herz and Bloch type spaces in bounded strongly pseudoconvex domains in $\mathbb{C}^n$. *Issues of analysis*, 2015, vol. 4, no. 1, pp. 73–94.
22. **Shamoyan R.F., Kurilenko S.M.** Bergman type integral operators on semiproducts of the tube domains over symmetric cones and multifunctionals analytic spaces. *Kragujevac Journal of Mathematics*, 2016, vol. 40, no. 2, pp.194–222.
23. **Shamoyan R.F., Mihic O.** Analytic classes on expanded disk and Bergman-type integral operators on them. *Journal of Inequalities and Applications*, 2009, art. ID 353801, 22 p.

24. **Antonenkova O.E., Shamoyan F.A.** The Cauchy transform of continuous linear functionals and projections on the weighted spaces of analytic functions. *Siberian Mathematical Journal*, 2005, vol. 46, no. 6, pp. 969–994.
25. **Gadbois S.** Mixed norm generalization of Bergman spaces and duality. *Proceedings of American Mathematical Society*, 1988, vol. 104, no. 4, pp. 1171–1180.
26. **Sehba B.F.** Bergman type operators in tube domains over symmetric cones. *Proceedings of Edinburg Mathematical Society*, 2009, vol. 52, no. 2, pp. 529–544.
27. **Gu D.** Bergman projections and duality in weighted mixed-norm spaces of analytic functions. *Michigan Mathematical Journal*, 1992, vol. 39, no. 1, pp. 71–84.
28. **Besov O.V., Il'in V.P., Nikol'skii S.M.** *Integral Representations of Functions and Imbedding Theorems*, vol. I, II. New York, Toronto, London, John Wiley & Sons, 1978, 1979.
29. **Nikol'skii S.M.** *Approximation of Functions of Several Variables and Imbedding Theorems*. Berlin, Heidelberg, New York, Springer-Verlag, 1975. 420 p.
30. **Shamoyan R.F., Mihic O.R.** On some new analytic function spaces in polyball. *Palestine Journal of Mathematics*, 2015, vol. 4, no. 1, pp. 105–107.
31. **Zhu K.** *Spaces of Holomorphic Functions in the Unit Ball*, New York, Springer-Verlag, 2004. 271 p.
32. **Bekolle D., Bonami A., Garrigos G., Ricci F., Sehba B.** Analytic Besov spaces and Hardy type inequalities in tube domains over symmetric cones. *Journal für die reine und angewandte Mathematik*, 2010, vol. 647, pp. 25–56.
33. **Shamoyan R.F., Povprits E.V.** Sharp theorems on traces in analytic spaces in tube domains over symmetric cones. *Zhurnal Sibirskogo federal'nogo universiteta* [Journal of Siberian Federal University. Series Mathematics and Physics], 2013, vol. 6, no. 4, pp. 527–538.
34. **Djrbashian A.E., Shamoyan F.A.** *Topics in the Theory of A^p_α-Spaces*, BSB B.G. Teubner, Leipzig, 1988.
35. **Xiao J.** *Geometric Q_p Functions*, Frontiers in Mathematics. Basel, Birkhäuser-Verlag, 2006.
36. **Xiao J.** *Holomorphic Q Classes*, Lecture Notes in Mathematics, 1767. Berlin, Heidelberg, Springer-Verlag, 2001. 104 p.
37. **Andersson M., Carlsson H.** Q_p spaces in strictly pseudoconvex domains. *Journal d'Analyse Mathematique*, 2001, vol. 84, no. 1, pp. 335–559.
38. **Shamoyan R.F.** On some problems connected with diagonal map in some spaces of analytic functions. *Mathematica Bohemica*, 2008, vol. 133, no. 4, pp. 351–366.
39. **Li S.Y., Luo W.** Characterizations for Besov spaces and applications, Part I. *Journal of Mathematical Analysis and Applications*, 2005, vol. 310, pp. 477–491.
40. **Karapetyan A.** Bounded projections in weighted function spaces in generalized unit disc. *Annales Polonici Mathematici*, 1995, vol. 462, no. 3, pp. 193–218.
41. **Bekolle D., Kagon A.** Reproducing properties and L^p estimates for Bergman projections in Siegel domains of type II. *Studia Mathematica*, 1995, vol. 15, no. 3, pp. 219–239.
42. **Yamaji S.** Composition operators on the Bergman space in minimal bounded domains. *Hiroshima Mathematical Journal*, 2003, vol. 43, pp. 107–127.
43. **Englis M., Hanninen T., Taskinen J.** Minimal L^∞ type spaces on strictly pseudoconvex domains on which Bergman projection is continuous. *Houston Journal of Mathematics*, 2006, vol. 32, no. 1, pp. 253–275.
44. **Li S.Y., Luo W.** Analysis on Besov spaces II: Embedding duality theorems. *Journal of Mathematical Analysis and Applications*, 2007, vol. 333, no. 2, pp. 1189–1202.
45. **Charpentier Ph., Dupain Y.** Estimates for the Bergman and Szego projections for pseudoconvex domains of finite type with locally diagonalizable Levi Form. *Publicacions Matemàtiques*, 2006, vol. 50, no. 2, pp. 413–446.

46. **Cho S.** A mapping property of the Bergman projection on certain pseudoconvex domains. *Tohoku Mathematical Journal*, 1996, vol. 48, no. 4, pp. 533–542.
47. **Ehsani D., Lieb I.** L^p estimates for the Bergman projection on pseudoconvex domains. *Mathematische Nachrichten*, 2008, vol. 281, no. 7, pp. 916–929.
48. **Shamoyan R.F., Mihic O.** Some remarks on weakly invertible functions in the unit ball and polydisk. *Iranian Journal of Mathematical Sciences and Informatics*, 2009, vol. 4, no. 1, pp. 43–55.
49. **Cho H.R., Lee J.** Inequalities for the integral means of holomorphic functions in the strongly pseudoconvex domain. *Communications of the Korean Mathematical Society*, 2005, vol. 20, no. 2, pp. 339–350.
50. **Ligocka E.** On the Forelli — Rudin construction and weighted Bergman projections. *Studia Mathematica*, 1989, vol. 94, pp. 257–272.
51. **Boas H.** Holomorphic reproducing kernels in Reinhardt domains. *Pacific Journal of Mathematics*, 1984, vol. 112, no. 2, pp. 273–292.
52. **Tkachenko (Makhina) N.M.** *Vesovye L^p otsenki analiticheskikh i garmonicheskikh funktsiy v odnosvyaznykh oblastyakh kompleksnoy ploskosti* [Weighted L^p estimates analytic and harmonic functions in a simply connected domains of complex plane. PhD Thesis]. Bryansk, 2009. (In Russ.).
53. **Tkachenko (Makhina) N.M., Shamoyan F.A.** The Hardy — Littlewood theorem and the operator of harmonic conjugate in some classes of simply connected domains with rectifiable boundary. *Journal of Mathematical Physics, Analysis, Geometry*, 2009, vol. 5, no. 2, pp. 192–210.
54. **Hahn K., Mitchell J.** Representation of linear functionals in H^p spaces over bounded symmetric domains in C^n. *Journal of Mathematical Analysis and Applications*, 1976, vol. 56, no. 2, pp. 379–396.
55. **Chen W.** Lipschitz spaces of holomorphic functions over bounded symmetric domains in C^n. *Journal of Mathematical Analysis and Applications*, 1981, vol. 81, no. 1, pp. 63–87.
56. **Nana C.** $L^{p,q}$-boundedness of Bergman projections in homogeneous Sigel domains of type II. *Journal of Fourier Analysis and Applications*, 2013, vol. 19, pp. 997–1019.
57. **Krantz S.G.** A direct connection between the Bergman and Szego projections. *Complex Analysis and Operators Theory*, 2014, vol. 8, pp. 571–579.
58. **Kalaj D., Markovic M.** Norm of the Bergman projection. *Mathematica Scandinavica*, 2014, vol. 115, no. 1, pp. 143–160.
59. **Bonami A., Garrigos G., Nana C.** $L^p - L^q$ estimates for Bergman projections in bounded symmetric domains of tube type. *Journal of Geometric Analysis*, 2014, vol. 24, no. 4, pp. 1737–1769.
60. **Bonami A., Nana C.** Some questions related to the Bergman projection in symmetric domains. *Advances in Pure and Applied Mathematics*, 2015, vol. 6, iss. 4. DOI:10.1515/apam-2015-5008.
61. **Legg A.R.** The Khavinson — Shapiro conjecture and the Bergman projection in one and several complex variables. *Computational Methods and Function Theory*, 2015, vol. 16, iss. 2, pp. 231–242.
62. **Peláez J.Á., Rättyä J.** *On the boundedness of the Bergman projection.* Available at: https://arxiv.org/abs/1501.03957, accessed 10.12.2016.

Accepted article received 06.05.2017
Corrections received 26.06.2017

Челябинский физико-математический журнал. 2017. Т. 2, вып. 2. С. 210–230.

ОГРАНИЧЕННЫЕ ИНТЕГРАЛЬНЫЕ ОПЕРАТОРЫ В НЕКОТОРЫХ ВЕСОВЫХ ПРОСТРАНСТВАХ АНАЛИТИЧЕСКИХ ФУНКЦИЙ В ДЕКАРТОВОМ ПРОИЗВЕДЕНИИ ОБЛАСТЕЙ И СВЯЗАННЫЕ РЕЗУЛЬТАТЫ

Р. Ф. Шамоян[a], **С. П. Максаков**[b]

Брянский государственный университет им. И. Г. Петровского, Брянск, Россия
[a]*rshamoyan@mail.ru*; [b]*msp222@mail.ru*

В обзорной статье собраны воедино полученные недавно результаты авторов и их коллег, касающиеся задачи существования и ограниченности интегральных проекторов типа Бергмана в пространствах аналитических функций нескольких переменных в различных многомерных областях и в декартовых произведениях таких областей. Задачи такого рода имеют многочисленные приложения в теории пространств аналитических функций нескольких комплексных переменных (теоремы о представлении функционалов, задача о мультипликаторах и т. д.). Также обсуждаются некоторые новые обобщения известных результатов. Кроме того, приведены и некоторые важные недавние результаты зарубежных авторов по этой проблеме.

Ключевые слова: *ограниченные операторы, пространства типа Бергмана, единичный круг, шар, полидиск, псевдовыпуклые и трубчатые области, аналитические пространства.*

Поступила в редакцию 06.05.2017
После переработки 26.06.2017

Сведения об авторах

Шамоян Роми Файзович, научный сотрудник кафедры математического анализа, Брянский государственный университет им. И. Г. Петровского, Брянск, Россия; e-mail: rshamoyan@mail.ru.

Максаков Серафим Павлович, студент, Брянский государственный университет им. И. Г. Петровского, Брянск, Россия; e-mail: msp222@mail.ru.