The polynomial differential equations of order 2 are considered. So we used correspondence map and normalizing transformation to get coefficients of Martinet — Ramis modulus.

Keywords: Martinet — Ramis modulus, saddlenode, central manifold.

Introduction

Consider a quadratic system of ordinary differential equations

\begin{align*}
\dot{p} &= p(1 - v) \\
\dot{v} &= v(p - v)
\end{align*}

It is a limit system for well known Jouanolou system [1] in a certain sense. System (1) has a saddlenode singularity at the origin.

Martinet — Ramis modulus \((C, \phi)\) (for saddlenode singular point) [2] are constructed by transformations reducing initial system to its (orbital) formal normal form. Solutions of system (1) with given initial conditions can be found as a series (with respect to initial condition). Using these solutions it is possible to find coefficients of the normalizing transformations, and then to determinate coefficients of the modulus.

As a result we get

\textbf{Theorem 1.} Let \((C, \phi)\) be Martinet-Ramis modulus for system (1). Then \(C = 0\), \(\phi(z) = z + 2\pi iz^2 + (2\pi i - 4\pi^2)z^3 + \ldots\)

\textbf{Corollary 1.} System (1) is not analytically orbital equivalent to its formal normal form.

1. Jouanolou’s systems

In the paper [1] there are considered a system of differential equations in \(\mathbb{C}P^2\) (in homogeneous coordinates)

\begin{align*}
\dot{a} &= b^n \\
\dot{b} &= c^n \\
\dot{c} &= a^n
\end{align*}

As it was shown [1], these systems (for \(n \geq 2\)) have not algebraic solutions (and it was first example of a system with such properties).

In affine coordinates \(x = a/b, y = b/c\) system (2) turn to

\[
\frac{dy}{dx} = \frac{1 - x^ny}{y^n - x^{n+1}}.
\]
Note that the rational transformation
\[ p = \frac{1}{y \cdot x^n}, \quad v = \frac{y^n}{x^{n+1}} \]
brings the last system to the quadratic form
\[ \frac{dv}{dp} = p[(1 - v) + \varepsilon(1 - p)] \]
\[ v[(p - v) + \varepsilon(1 - v)], \quad (3) \]
where \( \varepsilon = 1/n \). The limit transition \( (\varepsilon \to 0) \) leads (3) to system (1). It is important to note that system (1) could be obtained from system (2) (for \( n = -1 \)).

2. Construction of Martinet — Ramis modulus.

**Formal normal form and its first integral**

System (1) has a saddlenode singularity at the origin. Analytic classification of such singular points was given in [2]; it uses functional modulus. This functional modulus can be constructed by the next scheme.

Let \( V \) be a vector field with saddlenode singularity at the origin and \( V_0 \) be its (orbital) formal normal form. Let \( \{v = 0\} \) be its separatrix, corresponding to the hyperbolic eigenvalue of the field linearization at the origin. Let \( V \) be holomorphic in the polydisk \( U = \{|v| < \varepsilon, |p| < \varepsilon\} \) and domain \( \hat{U} \) be obtained from the polydisk \( U \) by the removing of the separatrix \( \{v = 0\} \).

As it was shown in [2], for sufficiently small \( \varepsilon \), the «cut» polydisk \( \hat{U} \) can be covered by a pair of domains \( U_\pm \), such that there exists a normalizing transformations \( H_\pm : U_\pm \to \mathbb{C}^2 \) \( (H_\pm \) transform phase curves of \( V \) into phase curves of \( V_0 \)). The intersection \( U_+ \cap U_- \) consists of two connected components: \( U_+ \cap U_- = W_+ \cup W_- \).

The transformation \( \Phi = H_+ \circ (H_-)^{-1}|_{U_+ \cap U_-} \) brings phase curves of formal normal form \( V_0 \) into itself. Values of the first integral \( J \) (standard for this formal normal form) numerate phase curves of the formal normal form. Therefore, the maps \( \Phi_\pm = \Phi|_{W_\pm} \) generate a pair of holomorphic functions \( \phi_\pm \) defined on domains \( w_\pm = J(H_-(W_\pm)) \) such that \( \phi_\pm \circ J = J \circ \Phi_\pm \).

As it was shown in [2] \( \phi_+(J) = J + C \), and \( \phi_-(J) \) is a holomorphic function in \( (\mathbb{C}, 0) \). The pair \( (C, \phi_-) \) (defined up to a linear transformation) is just the functional modulus of Martinet — Ramis.

3. Variations of solutions of system (1)

**with respect to the initial conditions**

To implement the program of construction of the Martinet — Ramis modulus, here we find a solution of system (1) (as a series with respect to the initial condition). Namely, we will search a solution \( p = P(v, p_0, v_0) \) of the differential equation
\[ \frac{dp}{dv} = \frac{p(1 - v)}{v(p - v)} \]
with the initial condition
\[ p(v_0) = p_0. \]
\[ (5) \]
Let
\[ p = \sum_{k=0}^{\infty} f_k(v)p_0^k, \]
\[ (6) \]
Substituting (6) in (4) we get a system of differential equations

\[
\begin{align*}
(-v + f_0) f'_0 &= \frac{1}{v} f_0, \\
-v f'_1 &= \frac{1}{v} f_1, \\
-v f'_2 + f_1 f'_1 &= \frac{1}{v} f_2, \\
-v f'_3 + f_1 f'_2 + f_2 f'_1 &= \frac{1}{v} f_3.
\end{align*}
\] (7)

From (5) we find initial conditions for the functions \(f_k\)

\[
f_1(v_0) = 1, \quad f_k(v_0) = 0, \quad k \neq 1.
\] (8)

Solving (7) and using (8), we get the functions

\[
\begin{align*}
f_0 &\equiv 0, \\
f_1 &= \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}}, \\
f_2 &= \frac{v}{v_0} e^{\frac{1}{v} - \frac{2}{v_0}} \left( \int_{v_0}^{v} \frac{e^{\frac{1}{S}}}{S} dS + e^{\frac{1}{v_0}} - e^{\frac{1}{v}} \right), \\
f_3 &= \frac{v}{2v_0^2} e^{\frac{1}{v} - \frac{3}{v_0}} \left[ \left( \int_{v_0}^{v} \frac{e^{\frac{1}{S}}}{S} dS \right)^2 - 4 \frac{1}{v_0} \int_{v_0}^{v} \frac{e^{\frac{1}{S}}}{S} dS + 4 e^{\frac{1}{v_0}} \right.
\end{align*}
\]

(9)

Remark 1. In all formulas the integration is done along some curve connecting points \(v_0\) and \(v\). The choice of these curves will be specified in section 5.

4. Correspondence map

Let us consider a transversal \(T_{v_0} = \{ v = v_0 \} \); as parameter on \(T_{v_0}\) we will use \(p\)-variable. We define a correspondence map \(\triangle_{v_0,v_1} : T_{v_0} \rightarrow T_{v_1}\) by the following way. Construct a solution \(p = P(v, p_0, v_0)\) of (4) with the initial condition \(p(v_0) = p_0\). These solution is holomorphic in a neighbourhood of \(v_0\). Let \(\gamma\) be a curve on the central manifold \(\{ p = 0, v \neq 0 \}\), connecting \(v_0\) and \(v_1\). Let us continue the function \(P(v, p_0, v_0)\) analytically along the curve \(\gamma\). And let \(p_1\) be the result of this continuation, \(p_1 = \triangle_{v_0,v_1}(p_0)\). Then, by definition, \(\triangle_{v_0,v_1}(p_0) = p_1\).

Example of a correspondence map for an orbital formal normal form.

Orbital formal normal form for (1) is

\[
\begin{align*}
\dot{p} &= p, \\
\dot{v} &= -v^2 \frac{1}{1-v}.
\end{align*}
\] (10)

Note that the function \(J = pv^{-1} e^{\frac{1}{v}}\) is constant on solutions of (10). Thus, correspondence map \(\tilde{f} : p_0 \mapsto p_1\) for (11) is defined from the equation

\[
J(v_0, p_0) = J(v_1, p_1).
\] (11)

Solving (11), we get

\[
p_1 = \triangle_{v_0}(p_0) = p_0 \frac{v_1}{v_0} e^{\frac{1}{v_1} - \frac{1}{v_0}}.
\]

Note that this map \(p_0 \mapsto p_1\) does not depend on \(\gamma\).
5. Normalizing transformation

Figure 2 illustrates this construction of normalizing transformation. A phase curve of the field $V$, going through the point $A$ of the transversal $T_{v_1} = \{ v = v_1 \}$, intersects the transversal $T_{v_0} = \{ v = v_0 \}$ at a point $A_0$. A phase curve of field $V_0$, going through the point $A_0$ of the transversal $T_{v_0} = \{ v = v_0 \}$, intersects the transversal $T_{v_1} = \{ v = v_1 \}$ in a point $\tilde{A}$. Then, by definition, $H(A) = \tilde{A}$. 
Here, the point $v_0$ is selected on the real line $\{\text{Im}(v) = 0, \text{Re}(v) > 0, p = 0\}$. For $v_1$, such that $\text{Re}(v_1) < 0$, the curve $\gamma$, that used in the construction of $H_-(v, p)$ ($H_+(v, p)$), have to go around above (below) of the deleted point $v = 0$ on the central manifold.

Finally, for $H_\pm$ we obtain $H_\pm : (v, p) \mapsto (v, \tilde{p}_\pm)$, where

$$\tilde{p}_\pm = \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}} P_\pm(v_0, p, v),$$

and $P_\pm$ are the functions, constructed in section 3, where the paths of integration in (9) are defined as above, with the correspondence of the sign.

### 6. Formulas for $\Phi$. Calculations of $\phi$-component

In the notation of section 6 for $\Phi$ we obtain formulas

$$\Phi(v, p_0) = (v, \tilde{p}),$$

where

$$\tilde{p} = \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}} P_+(v_0, \tilde{p}_0, v),$$

and

$$\tilde{p}_0 = P_-(v, p_0 e^{\frac{1}{v_0} - \frac{1}{v}}).$$

From section 2

$$\phi_-(J) = J \circ \Phi(v_0, p_0) = \tilde{p}v_0^{-1} e^{-\frac{1}{v_0}} =$$

$$= v_0^{-1} e^{-\frac{1}{v_0} - \frac{v}{v_0}} e^{\frac{1}{v} - \frac{1}{v_0}} P_+(v_0, P_- (v, p_0 e^{\frac{1}{v_0} - \frac{1}{v}}, v_0), v).$$

Substituting (9) into (12)–(14), from (15) we get

$$\phi(J) = J + 2\pi i J^2 + (2\pi i - 4\pi^2) J^3 + \ldots$$

Note that the transition functions $\Phi$ constructed here do not coincide with ones from section 2, but are conjugated with them. It means that the first coefficients of $\phi$ are found correctly. The proof of the theorem is finished.

The corollary is a sequence of the nontriviality of $\phi$-component of the modulus [2].

### References


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МОДУЛИ МАРТИНЕ — РАМИСА
ДЛЯ ОДНОЙ КВАДРАТИЧНОЙ СИСТЕМЫ
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В работе рассматривается полиномиальное дифференциальное уравнение степени 2. С помощью отображения соответствия и нормализующего преобразования получены коэффициенты модулей Мартине — Рамиса.

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