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MARTINET — RAMIS MODULUS FOR A QUADRATIC SYSTEM

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The polynomial differential equations of order 2 are considered. So we used correspondence map and normalizing transformation to get coefficients of Martinet — Ramis modulus.

Keywords: *Martinet — Ramis modulus, saddlenode, central manifold.*

Introduction

Consider a quadratic system of ordinary differential equations

$$\begin{cases} \dot{p} = p(1-v) \\ \dot{v} = v(p-v) \end{cases} \tag{1}$$

It is a limit system for well known Jouanolou system [1] in a certain sense. System (1) has a saddlenode singularity at the origin.

Martinet — Ramis modulus (C, ϕ) (for saddlenode singular point) [2] are constructed by transformations reducing initial system to its (orbital) formal normal form. Solutions of system (1) with given initial conditions can be found as a series (with respect to initial condition). Using these solutions it is possible to find coefficients of the normalizing transformations, and then to determinate coefficients of the modulus.

As a result we get

Theorem 1. *Let (C, ϕ) be Martinet-Ramis modulus for system (1). Then $C = 0$, $\phi(z) = z + 2\pi i z^2 + (2\pi i - 4\pi^2)z^3 + \dots$*

Corollary 1. *System (1) is not analytically orbital equivalent to its formal normal form.*

1. Jouanolou's systems

In the paper [1] there are considered a system of differential equations in $\mathbb{C}P^2$ (in homogeneous coordinates)

$$\begin{cases} \dot{a} = b^n \\ \dot{b} = c^n \\ \dot{c} = a^n \end{cases} \tag{2}$$

As it was shown [1], these systems (for $n \geq 2$) have not algebraic solutions (and it was first example of a system with such properties).

In affine coordinates $x = a/b$, $y = b/c$ system (2) turn to

$$\frac{dy}{dx} = \frac{1 - x^n y}{y^n - x^{n+1}}.$$

Note that the rational transformation

$$p = \frac{1}{y \cdot x^n}, \quad v = \frac{y^n}{x^{n+1}}$$

brings the last system to the quadratic form

$$\frac{dv}{dp} = \frac{p[(1-v)+\varepsilon(1-p)]}{v[(p-v)+\varepsilon(1-v)]}, \tag{3}$$

where $\varepsilon = 1/n$. The limit transition ($\varepsilon \to 0$) leads (3) to system (1). It is important to note that system (1) could be obtained from system (2) (for $n = -1$).

2. Construction of Martinet — Ramis modulus. Formal normal form and its first integral

System (1) has a saddlenode singularity at the origin. Analytic classification of such singular points was given in [2]; it uses functional modulus. This functional modulus can be constructed by the next scheme.

Let V be a vector field with saddlenode singularity at the origin and V_0 be its (orbital) formal normal form. Let $\{v = 0\}$ be its separatrix, corresponding to the hyperbolic eigenvalue of the field linearization at the origin. Let V be holomorphic in the polydisk $U = \{|v| < \varepsilon, |p| < \varepsilon\}$ and domain $\overset{\circ}{U}$ be obtained from the polydisk U by the removing of the separatrix $\{v = 0\}$.

As it was shown in [2], for sufficiently small ε, the «cut» polydisk $\overset{\circ}{U}$ can be covered by a pair of domains $U_\pm$, such that there exists a normalizing transformations $H_\pm : U_\pm \to \mathbb{C}^2$ ($H_\pm$ transform phase curves of V into phase curves of V_0). The intersection $U_+ \cap U_-$ consists of two connected components: $U_+ \cap U_- = W_+ \cup W_-$.

The transformation $\Phi = H_+ \circ (H_-)^{-1}|_{U_+ \cap U_-}$ brings phase curves of formal normal form V_0 into itself. Values of the first integral J (standard for this formal normal form) numerate phase curves of the formal normal form. Therefore, the maps $\Phi_\pm = \Phi|_{W_\pm}$ generate a pair of holomorphic functions $\phi_\pm$ defined on domains $w_\pm = J(H_-(W_\pm))$ such that $\phi_\pm \circ J = J \circ \Phi_\pm$.

As it was shown in [2] $\phi_+(J) = J + C$, and $\phi_-(J)$ is a holomorphic function in $(\mathbb{C}, 0)$. The pair (C, ϕ_-) (defined up to a linear transformation) is just the functional modulus of Martinet — Ramis.

3. Variations of solutions of system (1) with respect to the initial conditions

To implement the program of construction of the Martinet — Ramis modulus, here we find a solution of system (1) (as a series with respect to the initial condition). Namely, we will search a solution $p = P(v, p_0, v_0)$ of the differential equation

$$\frac{dp}{dv} = \frac{p(1-v)}{v(p-v)} \tag{4}$$

with the initial condition

$$p(v_0) = p_0. \tag{5}$$

Let

$$p = \sum_{k=0}^{\infty} f_k(v) p_0^k. \tag{6}$$

Substituting (6) in (4) we get a system of differential equations

$$\begin{gathered}(-v+f_0)f_0' = \tfrac{1-v}{v}f_0,\\ -vf_1' = \tfrac{1-v}{v}f_1,\\ -vf_2' + f_1f_1' = \tfrac{1-v}{v}f_2,\\ -vf_3' + f_1f_2' + f_2f_1' = \tfrac{1-v}{v}f_3.\end{gathered} \tag{7}$$

From (5) we find initial conditions for the functions f_k

$$f_1(v_0) = 1, \quad f_k(v_0) = 0,\ k \neq 1. \tag{8}$$

Solving (7) and using (8), we get the functions

$$f_0 \equiv 0,$$

$$f_1 = \frac{v}{v_0}e^{\frac{1}{v}-\frac{1}{v_0}},$$

$$f_2 = \frac{v}{v_0^2}e^{\frac{1}{v}-\frac{2}{v_0}}\left(\int_{v_0}^{v}\frac{e^{\frac{1}{S}}}{S}ds + e^{\frac{1}{v}} - e^{\frac{1}{v_0}}\right), \tag{9}$$

$$f_3 = \frac{v}{2v_0^3}e^{\frac{1}{v}-\frac{3}{v_0}}\left[\left(\int_{v_0}^{v}\frac{e^{\frac{1}{S}}}{S}dS\right)^2 - 4e^{\frac{1}{v_0}}\int_{v_0}^{v}\frac{e^{\frac{1}{S}}}{S}dS + 4e^{\frac{1}{v}}\int_{v_0}^{v}\frac{e^{\frac{1}{S}}}{S}dS + 3e^{\frac{2}{v}} + 2\int_{v_0}^{v}\frac{e^{\frac{1}{S}}}{S}dS + \right.$$
$$\left. + e^{\frac{2}{v_0}} - 4e^{\frac{1}{v}+\frac{1}{v_0}}\right].$$

Remark 1. In all formulas the integration is done along some curve connecting points v_0 and v. The choice of these curves will be specified in section 5.

4. Correspondence map

Let us consider a transversal $T_{v_0} = \{v = v_0\}$; as parameter on T_{v_0} we will use p-variable. We define a correspondence map $\triangle^{\gamma}_{v_0,v_1} : T_{v_0} \to T_{v_1}$ by the following way. Construct a solution $p = P(v, p_0, v_0)$ of (4) with the initial condition $p(v_0) = p_0$. These solution is holomorphic in a neighbourhood of v_0. Let γ be a curve on the central manifold $\{p = 0, v \neq 0\}$, connecting v_0 and v_1. Let us continue the function $P(v, p_0, v_0)$ analytically along the curve γ. And let p_1 be the result of this continuation, $p_1 = \triangle_{\gamma}P(v_1, p_0, v_0)$. Then, by definition, $\triangle^{\gamma}_{v_0,v_1}(p_0) = p_1$.

Example of a correspondence map for an orbital formal normal form.

Orbital formal normal form for (1) is

$$\begin{cases} \dot{p} = p, \\ \dot{v} = -\frac{v^2}{1-v}. \end{cases} \tag{10}$$

Note that the function $J = pv^{-1}e^{-\frac{1}{v}}$ is constant on solutions of (10). Thus, correspondence map $\tilde{f} : p_0 \mapsto p_1$ for (11) is defined from the equation

$$J(v_0, p_0) = J(v_1, p_1). \tag{11}$$

Solving (11), we get

$$p_1 = \triangle^{v_0}_{\gamma}(p_0) = p_0\frac{v_1}{v_0}e^{\frac{1}{v_1}-\frac{1}{v_0}}.$$

Note that this map $p_0 \mapsto p_1$ does not depend on γ.

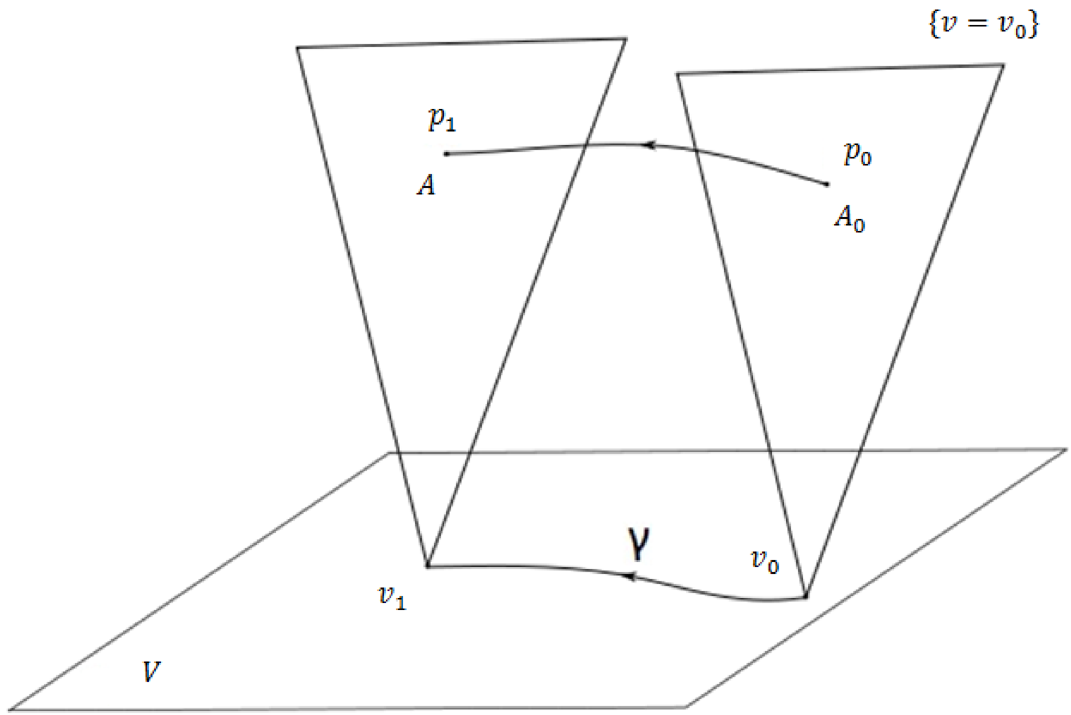

Fig. 1. Correspondence map for V

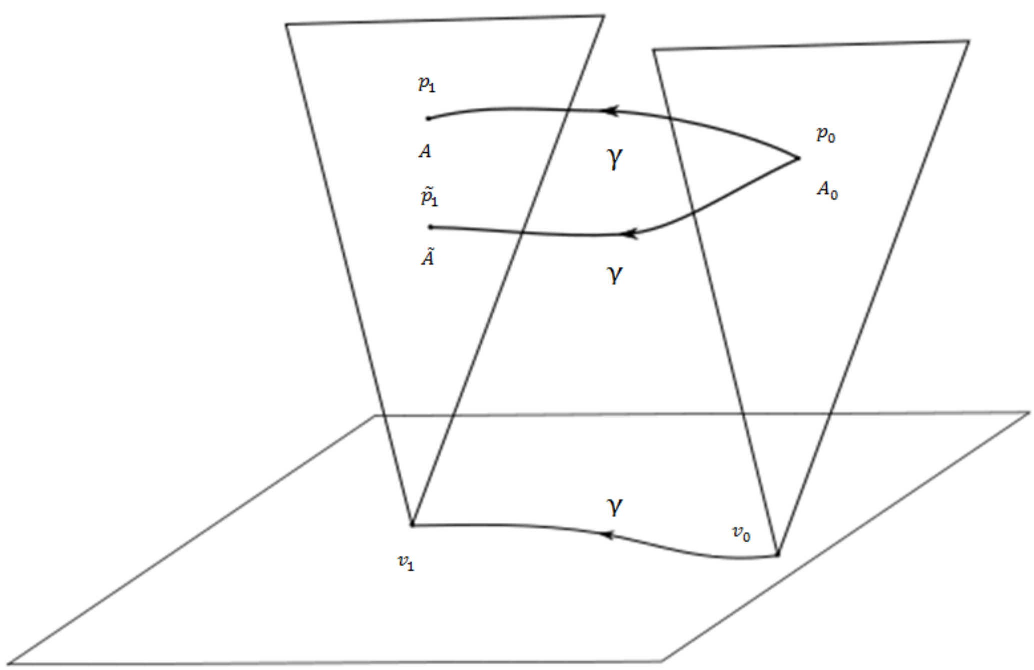

Fig. 2. Normalizing transformation

5. Normalizing transformation

Figure 2 illustrates this construction of normalizing transformation. A phase curve of the field V, going through the point A of the transversal $T_{v_1} = \{v = v_1\}$, intersects the transversal $T_{v_0} = \{v = v_0\}$ at a point A_0. A phase curve of field V_0, going through the point A_0 of the transversal $T_{v_0} = \{v = v_0\}$, intersects the transversal $T_{v_1} = \{v = v_1\}$ in a point $\tilde{A}$. Then, by definition, $H(A) = \tilde{A}$.

Here, the point v_0 is selected on the real line $\{\mathrm{Im}(v) = 0, \mathrm{Re}(v) > 0, p = 0\}$. For v_1, such that $\mathrm{Re}(v_1) < 0$, the curve γ, that used in the construction of $H_-(v,p)$ ($H_+(v,p)$), have to go around above (below) of the deleted point $v = 0$ on the central manifold.

Finally, for $H_\pm$ we obtain $H_\pm : (v,p) \mapsto (v, \tilde{p}_\pm)$, where

$$\tilde{p}_\pm = \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}} P_\pm(v_0, p, v),$$

and $P_\pm$ are the functions, constructed in section 3, where the paths of integration in (9) are defined as above, with the correspondence of the sign.

6. Formulas for Φ. Calculations of ϕ-component

In the notation of section 6 for Φ we obtain formulas

$$\Phi(v, p_0) = (v, \tilde{p}), \tag{12}$$

where

$$\tilde{p} = \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}} P_+(v_0, \tilde{p}_0, v), \tag{13}$$

and

$$\tilde{p}_0 = P_-\left(v, p_0 \frac{v_0}{v} e^{\frac{1}{v_0} - \frac{1}{v}, v_0}\right). \tag{14}$$

From section 2

$$\phi_-(J) = J \circ \Phi(v_0, p_0) = \hat{p} v_0^{-1} e^{-\frac{1}{v_0}} =$$
$$= v_0^{-1} e^{-\frac{1}{v_0}} \frac{v}{v_0} e^{\frac{1}{v} - \frac{1}{v_0}} P_+\left(v_0, P_-(v, p_0 \frac{v_0}{v} e^{\frac{1}{v_0} - \frac{1}{v}}, v_0), v\right). \tag{15}$$

Substituting (9) into (12)–(14), from (15) we get

$$\phi(J) = J + 2\pi i J^2 + (2\pi i - 4\pi^2) J^3 + \ldots$$

Note that the transition functions Φ constructed here do not coincide with ones from section 2, but are conjugated with them. It means that the first coefficients of ϕ are found correctly. The proof of the theorem is finished.

The corollary is a sequence of the nontriviality of ϕ-component of the modulus [2].

References

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МОДУЛИ МАРТИНЕ — РАМИСА ДЛЯ ОДНОЙ КВАДРАТИЧНОЙ СИСТЕМЫ

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Ключевые слова: *модуль Мартине —Рамиса, седлоузел, центральное многообразие.*

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