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c-ALMOST PERIODIC TYPE DISTRIBUTIONS

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We introduce and systematically analyze various classes of c-almost periodic type distributions and asymptotically c-almost periodic type distributions with values in complex Banach spaces. We provide an interesting application in the study of existence of asymptotically c-almost periodic type solutions for a class of ordinary differential equations in the distributional spaces.

Keywords: *c-almost periodic type distribution, asymptotically c-almost periodic type distribution, generalized function, ordinary differential equation, Banach space.*

1. Introduction and preliminaries

The concept of almost periodicity was introduced by H. Bohr [1] around 1924–1926 and later generalized by many other mathematicians. Let $I = \mathbb{R}$ or $I = [0, \infty)$, let $(X, \|\cdot\|)$ denote a complex Banach space, and let $f : I \to X$ be a continuous function. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t + \tau) - f(t)\| \leq \epsilon$, $t \in I$. The set of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I : X)$; see [2–11] and references cited therein for more details on the subject.

Let $\omega > 0$ and $c \in \mathbb{C}\backslash\{0\}$. The class of (ω, c)-periodic functions has been introduced and investigated by E. Alvarez, A. Gómez and M. Pinto in [12] (see also the research article [13] by E. Alvarez, S. Castillo and M. Pinto; in a series of our recent research studies, M.T. Khalladi, M. Kostić, A. Rahmani and D. Velinov have introduced and investigated generalized (ω, c)-almost periodic type functions). Let us recall that a continuous function $f : I \to X$ is said to be c-periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is (ω, c)-periodic, i. e., $f(x+\omega) = cf(x)$ for all $x \in I$ (here, $c \in \mathbb{C}\backslash\{0\}$). The space consisting of all (ω, c)-periodic functions and the space consisting of all c-periodic functions are denoted by $P_{\omega,c}(I : X)$ and $P_c(I : X)$, respectively; if $c = -1$, then we also say that a (-1)-periodic function is anti-periodic. The space $P_c(I : X)$ generalizes the space of Bloch (p, k)-periodic functions; let us recall that a bounded continuous function $f : I \to X$ is said to be Bloch (p, k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k (here, $p > 0$ and $k \in \mathbb{R}$) if and only if

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$f(x+p) = e^{ikp}f(x)$, $x \in I$. On the other hand, any anti-periodic function is Bloch (p,k)-periodic with certain real parameters $p > 0$ and $k \in \mathbb{R}$. It should be recalled that any Bloch (p,k)-periodic function is almost periodic [14] but not necessarily periodic since the function $f(x) := e^{ix} + e^{i(\sqrt{2}-1)x}$, $x \in \mathbb{R}$ is Bloch (p,k)-periodic with $p = 2\pi + \sqrt{2}\pi$ and $k = \sqrt{2} - 1$ but not periodic. For more details about (generalized) Bloch (p,k)-periodic functions, see the research articles [15] by M. Hasler and [16] by M.F. Hasler, G.M. N'Guérékata.

In [17], M.T. Khalladi, M. Kostić, M. Pinto, A. Rahmani and D. Velinov have analyzed the notion which depends on only one parameter, $c \in \mathbb{C}\backslash\{0\}$. More precisely, in this paper, the authors have introduced and analyzed various notions of c-almost periodicity for a continuous function $f : I \to X$:

(i) we call a number $\tau > 0$ a (ε, c)-period for $f(\cdot)$ if and only if $\|f(t+\tau) - cf(t)\| \leq \varepsilon$ for all $t \in I$ ($\varepsilon > 0$); by $\theta_c(f,\varepsilon)$ we denote the set of all (ε,c)-periods for $f(\cdot)$. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\varepsilon > 0$ the set $\theta_c(f,\varepsilon)$ is relatively dense in $[0,\infty)$. The space of all c-almost periodic functions from I into X will be denoted by $AP_c(I : X)$;

(ii) $f(\cdot)$ is called c-uniformly recurrent if and only if there exists a strictly increasing sequence (a_n) of positive real numbers such that $\lim_{n\to+\infty} a_n = +\infty$ and

$$\lim_{n\to+\infty} \left\| f(\cdot + a_n) - cf(\cdot) \right\|_\infty = 0.$$

If $c = -1$, then $f(\cdot)$ is said to be uniformly anti-recurrent. The space of c-uniformly recurrent functions from I into X will be denoted by $UR_c(I : X)$;

(iii) set $\mathbb{S} := \mathbb{N}$ if $I = [0,\infty)$ and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$; then it is said that:

(a) $f(\cdot)$ is semi-c-periodic (of type 1) if and only if

$$\forall\varepsilon > 0 \quad \exists\omega > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \left\| f(x+m\omega) - c^m f(x) \right\| \leq \varepsilon;$$

the space of all semi-c-periodic functions will be denoted by $SAP_c(I : X)$;

(b) $f(\cdot)$ is semi-c-periodic of type 2 if and only if

$$\forall\varepsilon > 0 \quad \exists\omega > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \left\| c^{-m} f(x+m\omega) - f(x) \right\| \leq \varepsilon;$$

(c) $f(\cdot)$ is semi-c-periodic of type 1_+ if and only if

$$\forall\varepsilon > 0 \quad \exists\omega > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \left\| f(x+m\omega) - c^m f(x) \right\| \leq \varepsilon;$$

(d) $f(\cdot)$ is semi-c-periodic of type 2_+ if and only if

$$\forall\varepsilon > 0 \quad \exists\omega > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \left\| c^{-m} f(x+m\omega) - f(x) \right\| \leq \varepsilon.$$

The space $P_c(I : X)$ is contained in the space $SAP_c(I : X)$, which is further contained in the space $AP_c(I : X)$. In [17], we have proved the following: Let $|c| = 1$, $i \in \{1,2\}$ and $f : I \to X$ be a continuous function. Then $f(\cdot)$ is semi-c-periodic of type i (i_+) if and only if $f(\cdot)$ is semi-c-periodic of type 1. We will also employ the following result whose proof will appear somewhere else.

Lemma 1. *Let $|c| \neq 1$, $i \in \{1,2\}$ and $f : I \to X$. Then $f(\cdot)$ is semi-c-periodic of type i (i_+) if and only if $f(\cdot)$ is c-periodic.*

A continuous function $f : I \to X$ is called asymptotically c-almost periodic (asymptotically c-uniformly recurrent, asymptotically semi-c-periodic) if and only if there are a c-almost periodic function (c-uniformly recurrent function, semi-c-periodic function) $g : \mathbb{R} \to X$ and a function $h \in C_0(I : X)$ such that $f(t) = g(t) + h(t)$,

$t \in I$; here, $C_0(I : X)$ denotes the vector space consisting of all continuous functions $f : I \to X$ such that $\lim_{|t|\to+\infty} \|f(t)\| = 0$. Suppose that $I = \mathbb{R}$. In this case, it is worth noting that the notion of asymptotical almost periodicity introduced above ($c = 1$) is different from the corresponding notion used in [18] (see also [10, Section 2.7]), where we have assumed that the decomposition $f(t) = g(t) + h(t)$ holds only for $t \geq 0$ as well as that $h \in C_0([0,\infty) : X)$ (cf. also [19, Remark 2.5]). In order to be more consistent henceforth, we will say that $f : \mathbb{R} \to X$ is half-asymptotically c-almost periodic (half-asymptotically c-uniformly recurrent, half-asymptotically semi-c-periodic) if and only if there are a c-almost periodic function (c-uniformly recurrent function, semi-c-periodic function) $g : \mathbb{R} \to X$ and a function $h \in C_0([0,\infty) : X)$ such that $f(t) = g(t) + h(t)$, $t \geq 0$.

If $c = 1$, then we also say that $f(\cdot)$ is ((half-)asymptotically) uniformly recurrent (((half-)asymptotically) semi-periodic, ((half-)asymptotically) almost periodic); if $c = -1$, then we also say that $f(\cdot)$ is ((half-)asymptotically) almost anti-periodic (((half-)asymptotically) uniformly anti-recurrent, ((half-)asymptotically) semi-anti-periodic). Note, if $f(\cdot)$ is c-almost periodic, then $f(\cdot)$ is almost periodic and therefore bounded (see [17]).

We will use the following lemma, which can be deduced with the help of [19, Theorem 2.6] and [9, Theorem 3.36, Theorem 3.47; pp. 97–98].

Lemma 2. *Suppose that the sequence ($f_n : \mathbb{R} \to X$) of asymptotically almost periodic functions (half-asymptotically almost periodic functions) converges uniformly to a function $f : \mathbb{R} \to X$. Then $f(\cdot)$ is asymptotically almost periodic (half-asymptotically almost periodic).*

The classes of scalar-valued bounded and almost periodic distributions have been introduced by L. Schwartz [20] and later extended to the vector-valued distributions by I. Cioranescu in [21]. On the other hand, the class of scalar-valued asymptotically almost periodic distributions has been introduced by I. Cioranescu in [22], while the notion of a vector-valued asymptotically almost periodic distribution has been analyzed by D.N. Cheban [23] following a different approach (cf. also I.K. Dontvi [5] and A. Halanay, D. Wexler [24]). For more details about the subject, we refer the reader to [25–32] as well as the recent research studies [33] by C. Bouzar, F. Z. Tchouar, [18] by M. Kostić and [34] by M. Kostić, S. Pilipović, D. Velinov.

As mentioned in the abstract, in this paper we introduce and investigate various classes of vector-valued c-almost periodic type distributions and vector-valued asymptotically c-almost periodic type distributions. The organization and main ideas of this paper, which is created as a certain continuation of studies [17] and [18], can be briefly described as follows. In Subsection 1.1, we remind ourselves of the basic definitions and results about vector-valued (asymptotically) almost periodic distributions (concerning original contributions of ours, we would like to say that, in this subsection, we introduce a new distributional space $B'_+(X)$ and provide a simple structural characterization of $B'_+(X)$ in Proposition 1). Section 2 introduces and thoroughly analyzes the above-mentioned classes of c-almost periodic type distributions; the main results of paper are Theorem 9 and Theorem 11. In the last section, we analyze the asymptotically c-almost periodic type solutions for the systems of ordinary differential equations in the distributional spaces.

1.1. Vector-valued (asymptotically) almost periodic distributions

Denote by $\mathcal{D}(X) = \mathcal{D}(\mathbb{R} : X)$ the Schwartz space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support in X. By $\mathcal{S}(X) = \mathcal{S}(\mathbb{R} : X)$ we denote the Schwartz space of all rapidly decreasing functions with values in X, and by $\mathcal{E}(X) = \mathcal{E}(\mathbb{R} : X)$ we denote the space of all infinitely differentiable functions with values in X; $\mathcal{D} \equiv \mathcal{D}(\mathbb{C})$, $\mathcal{S} \equiv \mathcal{S}(\mathbb{C})$ and $\mathcal{E} \equiv \mathcal{E}(\mathbb{C})$. The spaces of all linear continuous mappings from $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}$ into X are denoted by $\mathcal{D}'(X)$, $\mathcal{S}'(X)$ and $\mathcal{E}'(X)$, respectively [20]; $\mathcal{D}_0$ stands for the subspace of $\mathcal{D}$ consisting of all functions with the support contained in $[0, \infty)$. If $T \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}$, then we define $T * \varphi \in \mathcal{E}(X)$ by $(T * \varphi)(x) := \langle T, \varphi(x - \cdot)\rangle$. If $f : \mathbb{R} \to X$, then we define $\check{f} : \mathbb{R} \to X$ by $\check{f}(t) := f(-t)$, $t \in \mathbb{R}$; for any $T \in \mathcal{D}'(X)$, we define $\check{T} \in \mathcal{D}'(X)$ by $\langle \check{T}, \varphi\rangle := \langle T, \check{\varphi}\rangle$, $\varphi \in \mathcal{D}$.

Let $1 \leq p \leq \infty$. By $\mathcal{D}_{L^p}(\mathbb{R} : X)$ we denote the vector space consisting of all infinitely differentiable functions $f : \mathbb{R} \to X$ such that $f^{(j)} \in L^p(\mathbb{R} : X)$ for all $j \in \mathbb{N}_0$. The Fréchet topology on $\mathcal{D}_{L^p}(\mathbb{R} : X)$ is induced by the following system of norms $\|f\|_k := \sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\mathbb{R})}$, $k \in \mathbb{N}$. If $X = \mathbb{C}$, then the above space is simply denoted by $\mathcal{D}_{L^p}$. The space of all linear continuous mappings $f : \mathcal{D}_{L^1} \to X$ is denoted by $\mathcal{D}'_{L^1}(X)$. Endowed with the strong topology, $\mathcal{D}'_{L^1}(X)$ becomes a complete locally convex space; $\mathcal{D}'_{L^1}(X)$ is a well known space of bounded X-valued distributions. In the sequel, we will use the fact that a vector-valued distribution $T \in \mathcal{D}'(X)$ is bounded if and only if the function $T * \varphi$ is bounded for all $\varphi \in \mathcal{D}$; see e. g., [21, Theorem 1.1].

Let $T \in \mathcal{D}'_{L^1}(X)$. Then the following assertions are equivalent [21]:

(i) $T * \varphi \in AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$;

(ii) there exist an integer $k \in \mathbb{N}$ and almost periodic functions $f_j(\cdot) : \mathbb{R} \to X$ $(0 \leq j \leq k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$ in the distributional sense.

We say that a bounded distribution $T \in \mathcal{D}'_{L^1}(X)$ is almost periodic if and only if T satisfies any of the above two equivalent conditions; if this is the case, then we know that the restriction of T to the space $\mathcal{S}$ is an X-valued tempered distribution [10]. By $B'_{AP}(X)$ we denote the space consisting of all almost periodic distributions.

Define the space of bounded distributions tending to zero at plus infinity as follows:

$$B'_{+,0}(X) := \Big\{T \in \mathcal{D}'_{L^1}(X)\,;\, \lim_{h \to +\infty} \langle T_h, \varphi\rangle = 0, \quad \varphi \in \mathcal{D}\Big\},$$

where $\langle T_h, \varphi\rangle := \langle T, \varphi(\cdot - h)\rangle$, $T \in \mathcal{D}'(X)$, $h > 0$. A bounded distribution $T \in \mathcal{D}'_{L^1}(X)$ is said to be asymptotically almost periodic if and only if there exist an almost periodic distribution $T_{ap} \in B'_{AP}(X)$ and a bounded distribution tending to zero at plus infinity $Q \in B'_{+,0}(X)$ such that $\langle T, \varphi\rangle = \langle T_{ap}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$. By $B'_{AAP}(X)$ we denote the vector space consisting of all asymptotically almost periodic distributions (see e. g., [18, Definition 1]).

Let $T \in \mathcal{D}'_{L^1}(X)$. Then we know that the following assertions are equivalent (see e.g., [18, Theorem 1]):

(i) $T \in B'_{AAP}(X)$;

(ii) the function $T * \varphi$ is half-asymptotically almost periodic for all $\varphi \in \mathcal{D}_0$;

(iii) the function $T * \varphi$ is half-asymptotically almost periodic for all $\varphi \in \mathcal{D}$;

(iv) there exist an integer $k \in \mathbb{N}$ and half-asymptotically almost periodic functions $f_j(\cdot) : \mathbb{R} \to X$ $(0 \leq j \leq k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$ on $[0, \infty)$, i. e.,

$$\langle T, \varphi\rangle = \sum_{j=0}^{k} (-1)^j \int_0^\infty \varphi^{(j)}(t) f_j(t)\, dt, \quad \varphi \in \mathcal{D}_0; \tag{1}$$

(v) there exists a sequence (T_n) of half-asymptotically almost periodic functions from $\mathcal{E}(X)$ such that $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'_{L^1}(X)$.

For the first time in the existing literature, we consider here the space

$$B'_+(X) := \Big\{T \in \mathcal{D}'_{L^1}(X)\,;\ \lim_{|h|\to+\infty}\langle T_h, \varphi\rangle = 0, \quad \varphi \in \mathcal{D}\Big\},$$

which is slightly different from the space $B'_{+,0}(X)$ used above. For example, the regular distribution determined by the locally integrable function $f : \mathbb{R} \to \mathbb{R}$, given by $f(t) := 1$ for $t \le 0$ and $f(t) := 0$ for $t > 0$, belongs to the space $B'_{+,0}(X)$ but not to the space $B'_+(X)$. Since for every fixed test function $\varphi \in \mathcal{D}$ and for every real number $h \in \mathbb{R}$ we have

$$\langle \check{T}, \varphi(\cdot - h)\rangle = \langle T, \varphi(-\cdot -h)\rangle = \langle T, \check{\varphi}(\cdot - h)\rangle,$$

it follows that $T \in B'_+(X)$ if and only if $T \in B'_{+,0}(X)$ and $\check{T} \in B'_{+,0}(X)$. Therefore, [22, Proposition 1] immediately implies the following result (see also [33, Proposition 10]):

Proposition 1. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*

(i) $T \in B'_+(X)$;

(ii) *the restrictions of functions $T * \varphi$ and $\check{T} * \varphi$ to the non-negative real axis belong to the space $C_0([0,\infty) : X)$ for all $\varphi \in \mathcal{D}$;*

(iii) *there exist an integer $k \in \mathbb{N}$ and functions $f_j \in C_0(\mathbb{R} : X)$ $(0 \le j \le k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$;*

(iv) *there exists a sequence (T_n) in $\mathcal{E}'(X)$ which converges to T for topology of $\mathcal{D}'_{L^1}(X)$.*

2. c-Almost periodic type distributions and asymptotically c-almost periodic type distributions

We start this section by introducing the following notion:

Definition 1. Let $T \in \mathcal{D}'(X)$ and $c \in \mathbb{C} \setminus \{0\}$.

(i) T is said to be a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution, (AP_c) $((UR_c), (SAP_c))$ distribution in short, if and only if $T*\varphi \in AP_c(\mathbb{R} : X)$ $(T*\varphi \in UR_c(\mathbb{R} : X),\ T*\varphi \in SAP_c(\mathbb{R} : X))$ for all $\varphi \in \mathcal{D}$. By $B'_{AP_c}(X)$ $(B'_{UR_c}(X), B'_{SAP_c}(X))$ we denote the space of all c-almost periodic (c-uniformly recurrent, semi-c-periodic) distributions;

(ii) T is said to be a (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) distribution if and only if the function $T * \varphi$ is (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) for all $\varphi \in \mathcal{D}$;

(iii) T is said to be a (half-)asymptotically $(\mathcal{D}_0, c)$-almost periodic ((half-)asymptotically $(\mathcal{D}_0, c)$-uniformly recurrent, (half-)asymptotically semi-$(\mathcal{D}_0, c)$-periodic) distribution if and only if the function $T * \varphi$ is (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) for all $\varphi \in \mathcal{D}_0$.

Remark 1. In [14, Definition 2], we have introduced the notion of a semi-Bloch k-periodic function $(k \in \mathbb{R})$. The class of semi-Bloch k-periodic distributions can be also introduced but we will skip all related details concerning this notion for simplicity.

All distribution spaces introduced in Definition 1 are closed under differentiation. It is also clear that, if $T \in \mathcal{D}'(X)$ belongs to any of the spaces introduced above, then the distribution αT belongs to the same space, where $\alpha \in \mathbb{C}$ and $\langle \alpha T, \varphi \rangle := \langle T, \alpha\varphi \rangle$, $\varphi \in \mathcal{D}$; but, if $c \neq 1$, then the spaces of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) are not closed under pointwise addition, which continues to hold for corresponding distribution spaces. Further on, since every c-almost periodic (semi-c-periodic) function is almost periodic and therefore bounded continuous, an application of [21, Theorem 1.1] shows that any c-almost periodic (semi-c-periodic) distribution is a bounded distribution. This is no longer true for the c-uniformly recurrent distributions because there exists an unbounded uniformly recurrent function [35] and therefore the regular distribution determined by this function is a uniformly recurrent distribution which is not a bounded distribution ($c = 1$).

We continue by stating the following:

Proposition 2. *Suppose that T is a c-uniformly recurrent distribution and $c \in \mathbb{C}\setminus\{0\}$ satisfies $|c| \neq 1$. Then $T \equiv 0$.*

Proof. By definition, we have $T * \varphi \in UR_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}$. Since $|c| \neq 1$, Proposition 2.6 from [17] yields that $T * \varphi \equiv 0$ for all $\varphi \in \mathcal{D}$. This immediately implies $T = 0$. □

A distribution $T \in \mathcal{D}'(X)$ is called c-periodic if and only if $T * \varphi \in P_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}$. Keeping in mind Lemma 1, we can similarly prove the following:

Proposition 3. *Suppose that T is a semi-c-periodic distribution and $c \in \mathbb{C}\setminus\{0\}$ satisfies $|c| \neq 1$. Then T is c-periodic.*

Keeping in mind Proposition 2 and Proposition 3, it seems reasonable to impose the following condition:

Blank Hypothesis. Unless stated otherwise, we will always assume henceforth that $c \in \mathbb{C}$ and $|c| = 1$.

Now we will state and prove the following proposition:

Proposition 4. *The following statements are equivalent:*

(i) $T \in B'_{AP_c}(X)$ $(T \in B'_{UR_c}(X),\ T \in B'_{SAP_c}(X))$;

(ii) $\check{T} \in \mathcal{D}'_{AP_{1/c}}(X)$ $(\check{T} \in \mathcal{D}'_{UR_{1/c}}(X),\ \check{T} \in \mathcal{D}'_{SAP_{1/c}}(X))$.

Proof. Clearly, it suffices to show that (i) implies (ii). We will do that only for c-almost periodicity. Let $\varphi \in \mathcal{D}$ be fixed; we need to show that $\check{T} * \varphi \in AP_{1/c}(\mathbb{R} : X)$. Keeping in mind [17, Proposition 2.7], it suffices to show that

$$\check{T} * \varphi = T \check{*} \check{\varphi}, \quad \varphi \in \mathcal{D}. \tag{2}$$

To prove this equality, fix a real number $t \in \mathbb{R}$. Then (2) follows from the next simple computations:

$$\big(\check{T} * \varphi\big)(t) = \big\langle \check{T}, \varphi(t - \cdot)\big\rangle = \big\langle T, \varphi(t \check{-} \cdot)\big\rangle = \langle T, \varphi(t + \cdot)\rangle,$$

$$T \check{*} \check{\varphi}(t) = \big(T * \check{\varphi}\big)(-t) = \big\langle T, \check{\varphi}(-t - \cdot)\big\rangle = \big\langle T, \varphi(t - \cdot)\big\rangle = \langle T, \varphi(t + \cdot)\rangle.$$

□

We continue by observing that [17, Proposition 2.8, Proposition 2.9] directly imply the following: if $T \in B'_{AP_c}(X)$ $(T \in B'_{UR_c}(X),\ T \in B'_{SAP_c}(X))$, then $\|T * \varphi\| : I \to [0, \infty)$ is almost periodic (uniformly recurrent, semi-periodic) for all $\varphi \in \mathcal{D}$ as well as $T \in B'_{AP_{c^l}}(X)$ $(T \in B'_{UR_{c^l}}(X),\ T \in B'_{SAP_{c^l}}(X))$ for any positive integer $l \in \mathbb{N}$.

Furthermore, [17, Corollary 2.10, Proposition 2.11, Proposition 2.12] directly imply the following:

(i) suppose that $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$, $(p,q)=1$ and $\arg(c)=p\pi/q$ and $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X)$, $T \in B'_{SAP_c}(X)$);

(a) if p is even, then $T \in B'_{AP}(X)$ ($T \in B'_{UR}(X)$, $T \in B'_{SAP}(X)$);

(b) if p is odd, then T is almost anti-periodic (uniformly anti-recurrent, semi-anti-periodic) distribution;

(ii) suppose that $\arg(c)/\pi \notin \mathbb{Q}$ and $T \in B'_{AP_c}(X)$, then $T \in B'_{AP_{c'}}(X)$ for all $c' \in S_1 \equiv \{z \in \mathbb{C} : |z|=1\}$;

(iii) suppose that $\arg(c)/\pi \in \mathbb{Q}$ and $T \in B'_{SAP_c}(X)$, then $T \in B'_{AP_{c'}}(X)$ for all $c' \in \{c^l : l \in \mathbb{N}\}$.

(iv) suppose that $\arg(c)/\pi \notin \mathbb{Q}$ and $T \in B'_{SAP_c}(X)$, then $T \in B'_{AP_{c'}}(X)$ for all $c' \in S_1$.

The following statements known for functions can be simply deduced for distributions, as well (see [17, Example 2.22] for more details):

(i) suppose that $c=1$. Then the set consisting of all c-almost periodic distributions is a vector space together with the usual operations, while the set consisting of c-uniformly recurrent distributions and the set consisting of semi-c-periodic distributions are not vector spaces together with the usual operations;

(ii) suppose that $c \neq 1$. Then the set consisting of all c-almost periodic (c-uniformly recurrent, semi-c-periodic) distributions is not a vector space together with the usual operations.

In [17, Example 2.23], we have considered the pointwise products of c-almost periodic type functions with the scalar-valued functions. It is worthwhile to mention that all established statements can be reformulated for the pointwise products of c-almost periodic type distributions with the scalar-valued infinitely differentiable functions; concerning Stepanov classes of c-almost periodic type functions, it should be noticed that [18, Proposition 1] continues to hold in our new framework. Details can be left to the interested readers.

We proceed by stating the following simple result:

Proposition 5. *Let $h \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$ and $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X)$, $T \in B'_{SAP_c}(X)$). Then:*

(i) *any translation T_h of $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X)$, $T \in B'_{SAP_c}(X)$) belongs to $B'_{AP_c}(X)$ ($B'_{UR_c}(X)$, $B'_{SAP_c}(X)$);*

(ii) *define the distribution Tb by $\langle Tb, \varphi\rangle := \langle T, \varphi(b\cdot)\rangle$, $\varphi \in \mathcal{D}$. Then $Tb \in B'_{AP_c}(X)$ ($Tb \in B'_{UR_c}(X)$, $Tb \in B'_{SAP_c}(X)$).*

Proof. We will prove the proposition only for c-almost periodicity. To show (i), suppose that $T \in B'_{AP_c}(X)$ and $\varphi \in \mathcal{D}$. Then we know that $T * \varphi \in AP_c(I : X)$. Due to the first part of [17, Theorem 2.13(iv)], the above implies that the function $x \mapsto \langle T, \varphi(x+h-\cdot)\rangle$, $x \in \mathbb{R}$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic). Now the conclusion follows from the calculation

$$\big(T_h * \varphi\big)(x) = \big\langle T_h, \varphi(x-\cdot)\big\rangle = \big\langle T_h, \varphi(x-\cdot)\big\rangle = \big\langle T, \varphi(x+h-\cdot)\big\rangle, \quad x \in \mathbb{R}.$$

To show (ii), define the test function $\varphi^b(\cdot)$ by $\varphi^b(t) := \varphi(bt)$, $t \in \mathbb{R}$. Then $G^b := T * \varphi^b \in AP_c(\mathbb{R} : X)$ and the required conclusion follows from the second part of [17, Theorem 2.13(iv)] and the calculation

$$\begin{aligned}\big(T^b * \varphi\big)(t) &= \big\langle T^b, \varphi(t-\cdot)\big\rangle = \big\langle T, \varphi(t-b\cdot)\big\rangle = \\ &= \big\langle T, \varphi(b((t/b)-\cdot))\big\rangle = \big\langle T, \varphi^b((t/b)-\cdot)\big\rangle = G^b(t/b), \quad t \in \mathbb{R}.\end{aligned}$$

□

The following result is a distributional analogue of [17, Proposition 2.18]:

Proposition 6. *Let* $T \in B'_{UR_c}(X)$ $(T \in B'_{SAP_c}(X))$ *and* $T \neq 0$. *Then* $T \notin B'_{+,0}(X)$.

Proof. Since $T \neq 0$, there exists $\varphi \in \mathcal{D}$ such that $T * \check{\varphi} \neq 0$. Clearly, $T * \check{\varphi}$ is a c-uniformly recurrent function (semi-c-periodic function), so that [17, Proposition 2.18] implies $T * \check{\varphi} \notin C_0(\mathbb{R} : X)$. Assume to the contrary that $T \in B'_{+,0}(X)$. Then we have

$$\big(T * \check{\varphi}\big)(t) = \langle T, \varphi(\cdot - t)\rangle \to 0 \text{ as } t \to +\infty,$$

which is a contradiction. □

The following result will be important in our further analyses:

Theorem 5. *Suppose that there exist an integer* $k \in \mathbb{N}$ *and* c*-almost periodic (*c*-uniformly recurrent, semi-*c*-periodic) functions* $f_j : \mathbb{R} \to X$ $(0 \leq j \leq k)$ *such that the function*

$$t \mapsto f(t) \equiv \big(f_0(t), \cdots, f_k(t)\big), \quad t \in \mathbb{R} \tag{3}$$

is c*-almost periodic (*c*-uniformly recurrent, semi-*c*-periodic). Define* $T := \sum_{j=0}^{k} f_j^{(j)}$. *Then* $T \in B'_{AP_c}(X)$ $(T \in B'_{UR_c}(X),\ T \in B'_{SAP_c}(X))$.

Proof. We will prove the theorem only for c-almost periodicity because the proofs for c-uniform recurrence and semi-c-periodicity are quite analogous. It is clear that $T \in \mathcal{D}'(X)$ as well as that (1) yields that for each $\varphi \in \mathcal{D}$ and $t \in \mathbb{R}$ we have:

$$(T * \varphi)(t) = \langle T, \varphi(t - \cdot)\rangle = \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \varphi^{(j)}(t - v) f_j(v)\, dv =$$

$$= \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \varphi^{(j)}(v) f_j(t - v)\, dv. \tag{4}$$

Let $\varepsilon > 0$ be given. Then the set $\theta_c(f, \varepsilon)$ is relatively dense in $[0, \infty)$; let $\tau \in \theta_c(f, \varepsilon)$ be arbitrary. Then the above computation shows that

$$\|(T * \varphi)(t + \tau) - c(T * \varphi)(t)\| \leq \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \Big|\varphi^{(j)}(v)\Big| \cdot \Big\|f_j(t + \tau - v) - c f_j(t - v)\Big\|\, dv \leq$$

$$\leq \varepsilon \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \Big|\varphi^{(j)}(v)\Big|\, dv, \quad \varphi \in \mathcal{D},\ t \in \mathbb{R},$$

which simply implies the required statement. □

The following counterexample demonstrates the fact that Theorem 5 does not generally hold if the function $f(\cdot)$, defined by (3), is not c-almost periodic (c-uniformly recurrent, semi-c-periodic); we will provide a direct non-trivial calculation showing this:

Example 1. Suppose that $c = -1$, $k = 1$, $f_0(t) = \cos t$ and $f_1(t) = \cos(2t)$ for all $t \in \mathbb{R}$. Then the function $f(\cdot)$, defined by (3), is not almost anti-periodic (see [36, Example 2.2(ii)]) and we have

$$\langle T, \varphi\rangle = \int_{-\infty}^{+\infty} \varphi(v) \cos v\, dv - \int_{-\infty}^{+\infty} \varphi'(v) \cos(2v)\, dv, \quad \varphi \in \mathcal{D}.$$

Due to (4), we have:

$$(T*\varphi)(t+\tau)+(T*\varphi)(t)=\int_{-\infty}^{+\infty}\varphi(v)[\cos(t+\tau-v)+\cos(t-v)]\,dv+$$
$$+\int_{-\infty}^{+\infty}\varphi'(v)[\cos(2(t+\tau-v))+\cos(2(t-v))]\,dv,\quad \varphi\in\mathcal{D},\ t\in\mathbb{R}.$$

Applying the partial integration, the above implies

$$(T*\varphi)(t+\tau)+(T*\varphi)(t)=$$
$$=\int_{-\infty}^{+\infty}\varphi(v)\Big[\cos(t+\tau-v)+\cos(t-v)-2\sin(2(t+\tau-v))-2\sin(2(t-v))\Big]\,dv,\quad (5)$$

for any $\varphi\in\mathcal{D}$ and $t\in\mathbb{R}$. Suppose that $\varphi\in\mathcal{D}$ is non-negative and its support belongs to the interval $[1/6,1/4]\subset[0,1/4]$ as well as that

$$0<\varepsilon<\int_0^{1/4}\varphi(v)\min((\sin v)/2,\cos v,2\sin(2v))\,dv \quad (6)$$

and

$$0<\varepsilon<\frac{2}{3}\int_0^{1/4}\varphi(v)\,dv\cdot\sin\Big(\frac{\pi}{4}-\frac{1}{8}\Big),\quad 0<\varepsilon<2\sin\frac{1}{12}\cdot\cos\frac{1}{4}\int_0^{1/4}\varphi(v)\,dv. \quad (7)$$

We will prove that the set $\theta_{-1}(T*\varphi,\varepsilon)$ is empty in the following, a rather technical, way. Suppose to the contrary that $\tau\in\theta_{-1}(T*\varphi,\varepsilon)$. Then (5) implies

$$\left|\int_0^{1/4}\varphi(v)\Big[\cos(t+\tau-v)+\cos(t-v)-2\sin(2(t+\tau-v))-2\sin(2(t-v))\Big]dv\right|<\varepsilon \quad (8)$$

for all $t\in\mathbb{R}$. Plugging $t=-\tau$, $t=\pi-\tau$ and $t=(\pi/2)-\tau$ in (8), we get:

$$\left|\int_0^{1/4}\varphi(v)\Big[\cos(v)+\cos(\tau+v)+2\sin(2v)+2\sin(2(\tau+v))\Big]\,dv\right|<\varepsilon, \quad (9)$$

$$\left|\int_0^{1/4}\varphi(v)\Big[-\cos(v)-\cos(\tau+v)+2\sin(2v)+2\sin(2(\tau+v))\Big]\,dv\right|<\varepsilon \quad (10)$$

and

$$\left|\int_0^{1/4}\varphi(v)\Big[\sin(v)+\sin(\tau+v)-2\sin(2v)-2\sin(2(\tau+v))\Big]\,dv\right|<\varepsilon, \quad (11)$$

respectively. Adding and subtracting of (9) and (10), we get:

$$\left|\int_0^{1/4}\varphi(v)\Big[\cos(v)+\cos(\tau+v)\Big]\,dv\right|<\varepsilon \quad (12)$$

and

$$\left|\int_0^{1/4}\varphi(v)\Big[\sin(2v)+\sin(2(\tau+v))\Big]\,dv\right|<\varepsilon/2, \quad (13)$$

respectively. Inserting (13) in (11), we get

$$\left|\int_0^{1/4}\varphi(v)\Big[\sin(v)+\sin(\tau+v)\Big]\,dv\right|<2\varepsilon. \quad (14)$$

Further on, there exist $k \in \mathbb{N} \cup \{-1, 0\}$ and $a \in [0, 2\pi)$ such that $\tau = (2k+1)\pi + a$. Then (12)–(14) gives

$$\left| \int_0^{1/4} \varphi(v) \Big[\cos(v) - \cos(v+a) \Big] \, dv \right| < \varepsilon, \tag{15}$$

$$\left| \int_0^{1/4} \varphi(v) \Big[\sin(2v) + \sin(2a+2v) \Big] \, dv \right| < \varepsilon/2 \tag{16}$$

and

$$\left| \int_0^{1/4} \varphi(v) \Big[\sin(v) - \sin(v+a) \Big] \, dv \right| < 2\varepsilon. \tag{17}$$

If $a \in [0, (\pi/2) - (1/4)]$, then $2a + 2v \in [0, \pi/2]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of number ε in (6) and the estimate (16); if $a \in [\pi, 2\pi - (1/4)]$, then $a+v \in [\pi, 2\pi]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of number ε in (6) and the estimate (17). Further on, if $a \in [\pi/2, \pi]$, then $a+v \in [\pi/2, 3\pi/2]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of number ε in (6) and the estimate (15). If $a \in [(\pi/2) - (1/4), \pi/2]$, then the estimates (15) and (17) imply

$$\left| \int_0^{1/4} \varphi(v) \sin(v + (a/2)) \cdot \sin(a/2) \, dv \right| < \varepsilon/2$$

and

$$\left| \int_0^{1/4} \varphi(v) \cos(v + (a/2)) \cdot \sin(a/2) \, dv \right| < \varepsilon,$$

respectively. By adding, we get

$$\left| \int_0^{1/4} \varphi(v) \big[\sin(v + (a/2)) + \cos(v + (a/2)) \big] \, dv \right| < \frac{3}{2}\varepsilon \cdot \big[\sin(a/2) \big]^{-1} \le \frac{3}{2 \sin\Big(\frac{\pi}{4} - \frac{1}{8} \Big)} \varepsilon,$$

which is a contradiction due to our choice of ε in (7) and the fact that $a + v \le (\pi+1)/4$ for all $v \in [0, 1/4]$ and therefore $\sin(v + (a/2)) + \cos(v + (a/2)) \ge 1$ for all $v \in [0, 1/4]$. Finally, if $a \in [2\pi - (1/4), 2\pi)$, then

$$\begin{aligned} \int_0^{1/4} \varphi(v) \Big[\sin(2v) + \sin(2a+2v) \Big] \, dv &= 2 \int_0^{1/4} \varphi(v) \sin(2v + a) \cdot \cos(a) \, dv \\ &= 2 \int_{1/6}^{1/4} \varphi(v) \sin(2v+a) \cdot \cos(a) \, dv \ge 2 \sin \frac{1}{12} \cdot \cos \frac{1}{4} \int_0^{1/4} \varphi(v) \, dv, \end{aligned}$$

which contradicts the second inequality in (7).

We continue by stating the following structural characterization of space $B'_{AP_c}(X)$ ($B'_{UR_c}(X)$, $B'_{SAP_c}(X)$):

Theorem 7. *Let $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X)$, $T \in B'_{SAP_c}(X)$) and let T be a bounded distribution. Then there exist an integer $p \in \mathbb{N}$ and a c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) function $F : \mathbb{R} \to X$ such that*

$$T = \sum_{j=0}^{p} (-1)^j \binom{p}{j} F^{(2j)} \tag{18}$$

in the distributional sense.

Proof. The proof essentially follows from the argumentation contained in the proof of [31, Theorem 1]; we will only outline the main details for c-almost periodicity because the proofs for c-uniform recurrence and semi-c-periodicity are quite analogous. Let us consider a fundamental solution G of the differential operator $(1-d^2/dx^2)^p$ for a certain sufficiently large natural number $p \in \mathbb{N}$ depending on T. By the proof of the above-mentioned theorem, we have that the convolution $F := T * G$ exists as a continuous function and (18) holds in the distributional sense; furthermore, there exists a sequence (φ_k) in $\mathcal{D}$ such that $\lim_{k\to+\infty}(T*\varphi_k)(t) = F(t)$, uniformly in $t \in \mathbb{R}$. Since for each integer $k \in \mathbb{N}$ the function $(T*\varphi_k)(\cdot)$ is c-almost periodic (apply also [21, Theorem 1.1] for c-uniform recurrence), an application of [17, Theorem 2.13(iii)] shows that $F(\cdot)$ is c-almost periodic, as well. This completes the proof. □

Now we are able to formulate and prove the following result:

Theorem 9. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*

(i) *we have $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X)$, $T \in B'_{SAP_c}(X)$);*

(ii) *there exist an integer $p \in \mathbb{N}$ and a c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) function $F : \mathbb{R} \to X$ such that (18) holds in the distributional sense;*

(iii) *there exist an integer $k \in \mathbb{N}$ and c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) functions $f_j : \mathbb{R} \to X$ $(0 \le j \le k)$ such that the function $f(\cdot)$, defined through (3), is c-almost periodic (c-uniformly recurrent, semi-c-periodic) and $T = \sum_{j=0}^{k} f_j^{(j)}$;*

(iv) *there exists a sequence (T_n) of c-almost periodic functions (bounded c-uniformly recurrent functions, semi-c-periodic functions) from $\mathcal{E}(X)$ such that $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'_{L^1}(X)$.*

Proof. The implication (i) ⇒ (ii) is proved in Theorem 7, while the implication (ii) ⇒ (iii) is trivial. The implication (iii) ⇒ (i) follows from Theorem 5; therefore, we have proved the equivalence of statements (i), (ii) and (iii). Their equivalence with (iv) essentially follows from the argumentation contained in the proof of [31, Proposition 7]; see also the proof of Theorem 11 below. □

As a direct consequence of Theorem 9 (see also [18, Remark 1(ii)]), we have the following:

Corollary 1. *Let (T_n) be a sequence in $B'_{AP_c}(X)$ ($B'_{UR_c}(X) \cap \mathcal{D}'_{L^1}(X)$, $B'_{SAP_c}(X)$), and let $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'_{L^1}(X)$. Then $T \in B'_{AP_c}(X)$ ($T \in B'_{UR_c}(X) \cap \mathcal{D}'_{L^1}(X)$, $T \in B'_{SAP_c}(X)$).*

For the sequel, we need the following definition:

Definition 2. Suppose that $T \in \mathcal{D}'(X)$.

(i) We say that T is an asymptotically c-almost periodic distribution of type 1 (asymptotically c-uniformly recurrent distribution of type 1, asymptotically semi-c-periodic distribution of type 1) if and only if there exist a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution $T_{apc} \in B'_{AP_c}(X)$, ($T_{urc} \in B'_{UR_c}(X)$, $T_{sapc} \in B'_{SAP_c}(X)$) and a distribution $Q \in B'_{+}(X)$ such that $\langle T, \varphi\rangle = \langle T_{apc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}$, ($\langle T, \varphi\rangle = \langle T_{urc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}$, $\langle T, \varphi\rangle = \langle T_{sapc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}$).

(ii) We say that T is an asymptotically $(\mathcal{D}_0, c)$-almost periodic distribution of type 1 (asymptotically $(\mathcal{D}_0, c)$-uniformly recurrent distribution of type 1, asymptotically semi-$(\mathcal{D}_0, c)$-periodic distribution of type 1) if and only if there exist a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution $T_{apc} \in B'_{AP_c}(X)$, ($T_{urc} \in B'_{UR_c}(X)$, $T_{sapc} \in B'_{SAP_c}(X)$) and a distribution $Q \in B'_{+}(X)$ such that $\langle T, \varphi\rangle = \langle T_{apc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$, ($\langle T, \varphi\rangle = \langle T_{urc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$, $\langle T, \varphi\rangle = \langle T_{sapc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$).

Remark 2. Concerning Definition 2 (ii), it should be noted that it is completely irrelevant whether we will write $Q \in B'_{+}(X)$ or $Q \in B'_{+,0}(X)$ here because any element $Q \in B'_{+,0}(X)$ can be extended to an element $\tilde{Q} \in B'_{+}(X)$ by the formula $\tilde{Q} := F \cdot Q$, where $F \in C^{\infty}(\mathbb{R})$ is any fixed function satisfying $F(t) = 1$ for all $t \geq 0$ and $F(t) = 0$ for all $t \leq -1$.

Remark 3. We note that the decompositions in Definition 2 are unique in the case of consideration of c-almost periodicity (semi-c-periodicity) because they are unique for almost periodicity [18].

Now we will prove the following asymptotical analogue of Theorem 9, which gives some new insights at the assertion of [18, Theorem 1] and [33, Theorem 2] (in the last mentioned theorem, C. Bouzar and F. Z. Tchouar have recently established a structural characterization for the space of asymptotically almost automorphic distributions following the approach of I. Cioranescu from [22] (see also [18, Theorem 2]); our novelty here is the use of approach obeyed in the proof of [31, Proposition 7], with a direct proof of implication (i) ⇒ (ii) and a new characterization (iii) for the class of vector-valued asymptotically almost automorphic distributions):

Theorem 11. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*

(i) *T is (half-)asymptotically $(\mathcal{D}_0, c)$-almost periodic ((half-)asymptotically semi-$(\mathcal{D}_0, c)$-periodic);*

(ii) *T is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic);*

(iii) *there exist an integer $p \in \mathbb{N}$ and a bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that* (18) *holds in the distributional sense;*

(iii)’ *there exist an integer $p \in \mathbb{N}$ and a bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that* (18) *holds in the distributional sense ((18) holds in the distributional sense on $[0, \infty)$);*

(iv) *there exist an integer $k \in \mathbb{N}$ and bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) functions $f_j : \mathbb{R} \to X$ $(0 \leq j \leq k)$ such that the function $f(\cdot)$, defined through* (3), *is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) and $T = \sum_{j=0}^{k} f_j^{(j)}$;*

(iv)’ *there exist an integer $k \in \mathbb{N}$ and bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) functions $f_j : \mathbb{R} \to X$ $(0 \leq j \leq k)$ such that the function $f(\cdot)$, defined through* (3), *is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) and $T = \sum_{j=0}^{k} f_j^{(j)}$ ($T = \sum_{j=0}^{k} f_j^{(j)}$ on $[0, \infty)$);*

(v) *T is an asymptotically c-almost periodic distribution of type* 1 *(asymptotically semi-c-periodic distribution of type* 1*), in the case of consideration of asymptotical c-almost periodicity (asymptotical semi-c-periodicity), resp. T is an asymptotically*

$(\mathcal{D}_0, c)$-almost periodic distribution of type 1 (*asymptotically semi-$(\mathcal{D}_0, c)$–periodic distribution of type* 1), *in the case of consideration of half-asymptotical c-almost periodicity* (*half-asymptotical semi-c-periodicity*);

(vi) *there exists a sequence (T_n) of bounded (half-)asymptotically c-almost periodic functions (bounded (half-)asymptotically semi-c-periodic functions) from $\mathcal{E}(X)$ such that $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'_{L^1}(X)$.*

Proof. We will prove the implication (i) ⇒ (ii) only for half-asymptotical $(\mathcal{D}_0, c)$-almost periodicity. Let $\varphi \in \mathcal{D}$ be given and let $\operatorname{supp}(\varphi) \subseteq [a, b]$. If $a \geq 0$, then $\varphi \in \mathcal{D}_0$ and therefore the function $T * \varphi$ is half-asymptotically c-almost periodic, as required. If $a < 0$, then we consider the function $\varphi_a(\cdot) := \varphi(\cdot + a) \in \mathcal{D}_0$. Since the convolution mapping is translation invariant, we have that the function $(T * \varphi)_a(\cdot) = (T * \varphi_a)(\cdot)$ is half-asymptotically c-almost periodic, so that there exist a c-almost periodic function $g : \mathbb{R} \to X$ and a function $h \in C_0([0,\infty) : X)$ such that $(T * \varphi)_a(t) = (T * \varphi_a)(t) = g(t) + h(t)$ for all $t \geq 0$. This implies $(T * \varphi)(t) = g(t - a) + h(t - a) := g_a(t) + h_a(t)$, $t \geq a$. It is clear that the restriction of function $h_a(\cdot)$ to the non-negative real axis belongs to the space $C_0([0,\infty) : X)$, so that the statement (ii) follows by applying [17, Theorem 2.13(iv)] with $I = [0,\infty)$ and the number a replaced therein with the number $-a > 0$. The implication (ii) ⇒ (iii) can be proved following the lines of proof of Theorem 9; we will use the same notation. As in the proof of the above-mentioned result, we have that $\lim_{k\to+\infty}(T * \varphi_k)(t) = F(t)$, uniformly in $t \in \mathbb{R}$; due to [21, Theorem 1.1], the function $F(\cdot)$ is bounded. In the newly arisen situation, the function $(T * \varphi_k)(\cdot)$ is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) for all integers $k \in \mathbb{N}$. Therefore, there exist a c-almost periodic function (semi-c-periodic function) $g_k : \mathbb{R} \to X$ and a function $h_k \in C_0(\mathbb{R} : X)$, resp. $h_k \in C_0([0,\infty) : X)$, such that $(T * \varphi_k)(t) = g_k(t) + h_k(t)$, $t \in \mathbb{R}$, resp. $(T * \varphi_k)(t) = g_k(t) + h_k(t)$, $t \geq 0$ ($k \in \mathbb{N}$). Since for each integer $k \in \mathbb{N}$ the function $g_k(\cdot)$ is almost periodic, the use of Lemma 2 yields that there exists an almost periodic function $g : \mathbb{R} \to X$ and a function $\phi \in C_0(\mathbb{R} : X)$, resp. $\phi \in C_0([0,\infty) : X)$, such that $F(t) = g(t) + \phi(t)$ for all $t \in \mathbb{R}$, resp. $F(t) = g(t) + \phi(t)$ for all $t \geq 0$. But, the argumentation contained in the proofs of [9, Theorem 3.36, Theorem 3.47; pp. 97–98] also shows that the sequence of functions (g_k) converges to the function $g(\cdot)$, uniformly on $\mathbb{R}$. Since for each integer $k \in \mathbb{N}$ the function $g_k(\cdot)$ is c-almost periodic (semi-c-periodic), an application of [17, Theorem 2.13(iii)] shows that the function $g(\cdot)$ is also c-almost periodic (semi-c-periodic). This implies (iii). The implications (iii) ⇒ (iv) ⇒ (iv)' are trivial. We will prove that (iv)' implies (v) only for half-asymptotical c-almost periodicity. It simply follows that there exist c-almost periodic functions $g_j : \mathbb{R} \to X$ and functions $h_j \in C_0([0,\infty) : X)$ $(0 \leq j \leq k)$ such that the function $t \mapsto (g_0(t), \cdots, g_k(t))$, $t \in \mathbb{R}$ is c-almost periodic as well as that $f_j(t) = g_j(t) + h_j(t)$ for all $t \geq 0$. Define $T_{apc} \in B'_{AP_c}(X)$ by

$$T_{apc}(\varphi) := \sum_{j=0}^{k} (-1)^j \int_{-\infty}^{+\infty} \varphi^{(j)}(v) g_j(v)\, dv, \quad \varphi \in \mathcal{D}$$

(see Theorem 5) and

$$Q(\varphi) := \sum_{j=0}^{k} (-1)^j \int_{-\infty}^{+\infty} \varphi^{(j)}(v) h_j^e(v)\, dv, \quad \varphi \in \mathcal{D},$$

where $h_j^e(\cdot)$ denotes the even extension of the function $h_j(\cdot)$ to the whole real axis. It is clear that we have $\langle T, \varphi\rangle = \langle T_{apc}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$. In order to see that $Q \in B'_+(X)$, it suffices to observe that, for every test function $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subseteq [a, b]$, we have that

$$\langle Q, \varphi(\cdot - h)\rangle := \sum_{j=0}^{k} \int_a^b \varphi^{(j)}(v) h_j^e(v+h)\, dv, \quad \varphi \in \mathcal{D},\ h \in \mathbb{R}$$

and therefore $\lim_{|h|\to+\infty}\langle Q, \varphi(\cdot - h)\rangle = 0$, $\varphi \in \mathcal{D}$. In order to see that (v) implies (i), it suffices to repeat verbatim the argumentation given in [18, Remark 2]. We will prove that (vi) implies (i) only for half-asymptotical c-almost periodicity. Using the argumentation contained in the proof of [31, Proposition 7], it suffices to show that, for every fixed function $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subseteq [0, b]$ and for every fixed bounded half-asymptotically c-almost periodic function $f : \mathbb{R} \to X$, the function $\varphi * f$ is bounded and half-asymptotically c-almost periodic. This is clear for boundedness; in order to see that the function $\varphi * f$ is half-asymptotically c-almost periodic, we can argue as follows. Let $g : \mathbb{R} \to X$ be a c-almost periodic function and let $h \in C_0([0,\infty) : X)$ such that $f(t) = g(t) + h(t)$ for all $t \geq 0$. Then we have

$$(\varphi * f)(t) = \int_{-\infty}^{+\infty} \varphi(s) g(t-s)\, ds + \int_0^b \varphi(s) h(t-s)\, ds, \quad t \geq b,$$

so that the final conclusion follows from the fact that the space consisting of all c-almost periodic functions is convolution invariant as well as that the function

$$t \mapsto (\varphi * f)(t) - \int_{-\infty}^{+\infty} \varphi(s) g(t-s)\, ds, \quad t \geq 0$$

belongs to the class $C_0([0,\infty) : X)$, which a simple consequence of the last equality. The implication (i) $\Rightarrow$ (vi) follows directly from the corresponding part of the proof of [31, Proposition 7]. Therefore, we have proved the equivalence of all statements (i)–(vi). Since (iii)' implies (iv)' and (iv)' implies (v), we have that (iii)' or (iv)' implies all other statements (i)–(vi). On the other hand, it is clear that (iii) implies (iii)', finishing the proof. □

Corollary 2. *Let (T_n) be a sequence of bounded (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) distributions, and let* $\lim_{n\to\infty} T_n = T$ *in* $\mathcal{D}'_{L^1}(X)$. *Then T is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic).*

Remark 4.

(i) It is worth noting that the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) and the equivalence (vi) $\Leftrightarrow$ (i) can be formulated for bounded (half-)asymptotically c-uniformly recurrent functions, but it is not clear how one can prove that (ii) implies (iii) in this framework.

(ii) Using the idea from the proof of implication (i) $\Rightarrow$ (ii) of Theorem 11, we may conclude that a distribution $T \in \mathcal{D}'(X)$ is c-periodic (c-almost periodic, c-uniformly recurrent, semi-c-periodic) if and only if the function $T * \varphi$ is c-periodic (c-almost periodic, c-uniformly recurrent, semi-c-periodic) for all $\varphi \in \mathcal{D}_0$.

(iii) If $c \neq 1$, then it is not clear how we can introduce and analyze the classes of c-almost automorphic functions and c-almost automorphic distributions.

3. An application

Let $n \in \mathbb{N}$, and let $A = [a_{ij}]_{1\leq i,j\leq n}$ be a given complex matrix such that $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Following the analysis of C. Bouzar and M.T. Khalladi [28], we will provide here a small application in the analysis of the existence of half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) solutions of equation

$$T' = AT + G, \quad T \in \mathcal{D}'(X^n) \text{ on } [0,\infty), \tag{19}$$

where G is a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) X^n-valued distribution. By a solution of (19), we mean any element $T \in \mathcal{D}'(X^n)$ such that (19) holds in the distributional sense on $[0,\infty)$. Since the spaces of half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distributions are not closed under the pointwise addition of functions, some obvious unpleasant difficulties occur in the case that $c \neq 1$. In the one-dimensional case, these difficulties can be overcomed, fortunately:

Theorem 13. *Suppose that F is a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distribution, $n = 1$ and $a_{11} = \lambda < 0$. Then there exists a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distributional solution of (19). Furthermore, any distributional solution T of (19) is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic).*

Proof. By Theorem 11, we know that there exist an integer $p \in \mathbb{N}$ and a bounded half-asymptotically c-almost periodic (bounded half-asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that (18) holds in the distributional sense, with T replaced with G therein. By the proof of [18, Theorem 4], given in the ultradistributional case, we get the existence of a positive integer $m \in \mathbb{N}$, continuous functions $F_j : [0,\infty) \to X$ $(0 \leq j \leq m)$ and a function $Q \in C_0([0,\infty) : X)$ such that any function $F_j(\cdot)$ has the form

$$F_j(t) = c_{1,j}(\lambda)F(t) + c_{2,j}(\lambda)\int_0^t e^{\lambda(t-s)}F(s)\,ds, \ t \geq 0,$$

for certain complex numbers $c_{1,j}(\lambda)$ and $c_{2,j}(\lambda)$ $(0 \leq j \leq m)$ and $T = Q + \sum_{j=0}^m F_j^{(j)}$ on $[0,\infty)$. By the proofs of [10, Proposition 2.6.11] and [8, Lemma 4.1] (see also [17, Proposition 2.32]), we have that the function $t \mapsto (F_0(t), \cdot\cdot\cdot, F_m(t))$, $t \geq 0$ is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) so that it can be uniquely extended to a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) function $t \mapsto (\tilde{F_0}(t), \cdot\cdot\cdot, \tilde{F_m}(t))$, $t \in \mathbb{R}$ due to [17, Proposition 2.25]. Define $T_0 := \sum_{j=0}^m \tilde{F}_j^{(j)}$ and $T_1 := Q_e$, where Q_e denotes the even extension of function $Q(\cdot)$ to the whole real axis. Then $T = T_0 + T_1$ on $[0,\infty)$, T is c-almost periodic (semi-c-periodic) and $T_1 \in B'_+(X)$, so that T is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic). The existence of solutions is proved in a similar fashion. □

Unfortunately, the use of [28, Lemma 1] and the arguments contained in the proof of [28, Theorem 3, pp. 117-118] do not enable us to extend Theorem 13 to the multi-dimensional case. Keeping in mind the proofs of [18, Theorem 4] and Theorem 13, we can only prove the following:

Theorem 15. *Let there exist an integer $m \in \mathbb{N}$ and half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) X^n-valued functions $G_j(\cdot)$ $(0 \leq j \leq m)$ such that $G = \sum_{j=0}^m G_j^{(j)}$ on $[0,\infty)$. Then there exists a solution T of (19) which has the same form as G; furthermore, any distributional solution T of (19) has the same form as G (with the meaning clear).*

We close the paper with the observation that the various classes of c-periodic type (ultra-)distributions and various classes of c-almost periodic (ultra-)distributions will be considered somewhere else.

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РАСПРЕДЕЛЕНИЯ c-ПОЧТИ ПЕРИОДИЧЕСКОГО ТИПА

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Вводятся и систематически анализируются различные классы распределений c-почти периодического типа и распределений асимптотически c-почти периодического типа со значениями в комплексных банаховых пространствах. Предложено интересное приложение для изучения существования решений асимптотически c-почти периодического типа для одного класса обыкновенных дифференциальных уравнений в пространствах распределений.

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