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A NOTE ON SEMILINEAR DEGENERATE RELAXATION EQUATIONS ASSOCIATED WITH ABSTRACT DIFFERENTIAL OPERATORS

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The main aim of this paper is to prove some new results on semilinear degenerate relaxation equations associated with abstract differential operators. In order to do that, we use the Fourier multiplier techniques.

Keywords: *abstract semilinear degenerate relaxation differential equation, Mittag-Leffler function, Caputo time-fractional derivative,* $(\alpha, \alpha, \overline{P_1(A)}, \overline{P_2(A)}, C)$*-resolvent family, abstract differential operator.*

1. Introduction and preliminaries

Considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in engineering, physics, chemistry, biology and other sciences. Fairly complete information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting [1–7]. We refer to [8–11] for the basic source of information on the abstract degenerate differential equations with integer order derivatives in the time variable. The analysis of abstract degenerate time-fractional differential equations has been initiated in a series of recent papers. In this paper, we consider the existence and uniqueness of mild solutions of the semilinear degenerate relaxation equation $(\mathrm{DFP})_{sl}$ below, thus continuing the researches raised by R.-N. Wang, D.-H. Chen, T.-J. Xiao [12], V. Keyantuo, C. Lizama, M. Warma [13].

We use the standard notation throughout the paper. Unless specified otherwise, we assume that E is a Banach space over the field of complex numbers. By $L(E)$ and $\|x\|$ we denote the space consisting of all continuous linear mappings from E into E and the norm of an element $x \in E$, respectively. Let $I = [0, \infty)$ or $I = [0, T]$ for some $T > 0$. By $C(I : E)$ we denote the space consisting of all continuous functions from I into E. Equipped with the norm $\| \cdot \|_{C([0,T]:E)} := \sup_{t \in [0,T]} \| \cdot (t)\|$, $C([0, T] : E)$ becomes a complex Banach space. If A is a linear operator acting on E, then the domain and range of A will be denoted by $D(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. If $C \in L(E)$ is injective, then the norm $\| \cdot \|_{R(C)} := \|C^{-1} \cdot \|$ turns $R(C)$ into a Banach space, which will be simply denoted by $[R(C)]$. Suppose that B is a closed linear operator with domain and range contained in E. Then we define the set $\rho_C(B, A)$ by $\rho_C(B, A) := \{\lambda \in \mathbb{C}\ ;\ \lambda B - A : D(A) \cap D(B) \to E$ is injective and $(\lambda B - A)^{-1} C \in L(E)\}$.

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Sometimes we use the following condition on a scalar-valued function $K(\cdot)$:

(P1) $K(\cdot)$ is Laplace transformable, i. e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that

$$\tilde{K}(\lambda) := \mathcal{L}(K)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} K(t)\,dt := \int_0^\infty e^{-\lambda t} K(t)\,dt$$

exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \beta$. Put $\mathrm{abs}(K) := \inf\{\mathrm{Re}\lambda : \tilde{K}(\lambda) \text{ exists}\}$.

By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform on $\mathbb{R}^n$ and its inverse transform, respectively. Given $\theta \in (0,\pi]$ and $d \in (0,1]$, define $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0\}$. Further on, $\lfloor \beta \rfloor := \sup\{n \in \mathbb{Z} : n \leq \beta\}$ and $\lceil \beta \rceil := \inf\{n \in \mathbb{Z} : \beta \leq n\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f*g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ ($\zeta > 0$, $t > 0$) and $g_0(t) :=$ the Dirac δ-distribution. For a number $\zeta > 0$ given in advance, the Caputo fractional derivative $\mathbf{D}_s^\zeta u$ [1; 4] is defined for those functions $u \in C^{\lceil \zeta \rceil -1}([0,\infty) : E)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil -1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0,\infty) : E)$, by

$$\mathbf{D}_s^\zeta u(s) := \frac{d^{\lceil \zeta \rceil}}{ds^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil -1} u^{(j)}(0) g_{j+1} \right) \right].$$

The Mittag-Leffler function $E_{\beta,\gamma}(z)$ ($\beta > 0$, $\gamma \in \mathbb{R}$) is defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

In this place, we assume that $1/\Gamma(\beta k + \gamma) = 0$ if $\beta k + \gamma \in -\mathbb{N}_0$. Set, for short, $E_\beta(z) := E_{\beta,1}(z)$, $z \in \mathbb{C}$. The asymptotic behaviour of the entire function $E_{\beta,\gamma}(z)$ is given in the following auxiliary lemma (cf. [1; 4] and references cited there for further information about the Mittag-Leffler functions):

Lemma 1. *Let $0 < \sigma < \pi/2$. Then, for every $z \in \mathbb{C} \setminus \{0\}$ and $l \in \mathbb{N} \setminus \{1\}$,*

$$E_{\beta,\gamma}(z) = \frac{1}{\beta} \sum_s Z_s^{1-\gamma} e^{Z_s} - \sum_{j=1}^{l-1} \frac{z^{-j}}{\Gamma(\gamma - \beta j)} + O\big(|z|^{-l}\big), \quad |z| \to \infty,$$

where Z_s is defined by $Z_s := z^{1/\beta} e^{2\pi i s/\beta}$ and the first summation is taken over all those integers s satisfying $|\arg(z) + 2\pi s| < \beta(\sigma + \pi/2)$.

The reader may consult monographs [4] and [6] for more details concerning abstract non-degenerate Volterra integro-differential equations in Banach and locally convex spaces. We can consider the C-well-posedness of the following abstract degenerate Volterra equation:

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds,\ t \in [0,\tau), \tag{1}$$

where $0 < \tau \leq \infty$, $t \mapsto f(t)$, $t \in [0,\tau)$ is an E-valued continuous mapping and $a \in L^1_{loc}[0,\tau)$, $a \neq 0$. Following the approach of T.-J. Xiao and J. Liang [14; 15], we have introduced the following definition.

Definition 1. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t) : D(B) \to E$ is a linear mapping ($t \geq 0$). Let $C \in L(E)$ be injective, and let $CA \subseteq AC$. Then the operator family $(R(t))_{t\geq 0}$ is said to be an exponentially bounded (a,k)-regularized C-resolvent family for (1) iff there exists $\omega \geq \max(0, \operatorname{abs}(a), \operatorname{abs}(k))$ such that the following holds:

(i) The mapping $t \mapsto R(t)x$, $t \geq 0$, is continuous for every fixed element $x \in D(B)$.

(ii) There exists $M \geq 1$ such that

$$\|R(t)x\| \leq Me^{\omega t}\|x\|, \quad x \in D(B),\ t \geq 0.$$

(iii) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ and

$$\tilde{k}(\lambda)\big(B - \tilde{a}(\lambda)A\big)^{-1}CBx = \int\limits_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in D(B).$$

If $k(t) = g_{r+1}(t)$ for some $r \geq 0$, then it is also said that $(R(t))_{t\geq 0}$ is an exponentially bounded r-times integrated (a, C)-regularized resolvent family for (1); an exponentially bounded 0-times integrated (a, C)-regularized resolvent family for (1) is also said to be an exponentially bounded (a, C)-regularized resolvent family for (1).

Assume that $n \in \mathbb{N}$ and iA_j, $1 \leq j \leq n$, are commuting generators of bounded C_0-groups on a Banach space E. Set $A := (A_1, \cdot\cdot\cdot, A_n)$, $A^\eta := A_1^{\eta_1} \cdot\cdot\cdot A_n^{\eta_n}$ for any $\eta = (\eta_1, \cdot\cdot\cdot, \eta_n) \in \mathbb{N}_0^n$, and denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Let $k = 1 + \lfloor n/2 \rfloor$. For every $\xi = (\xi_1, \cdot\cdot\cdot, \xi_n) \in \mathbb{R}^n$ and $u \in \mathcal{F}L^1(\mathbb{R}^n) = \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$, we set $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$, $(\xi, A) := \sum_{j=1}^n \xi_j A_j$ and

$$u(A)x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi)e^{-i(\xi,A)}x\,d\xi,\ x \in E.$$

Then $u(A) \in L(E)$, $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and there exists a finite constant $M \geq 1$ such that

$$\|u(A)\| \leq M\big\|\mathcal{F}^{-1}u\big\|_{L^1(\mathbb{R}^n)},\ u \in \mathcal{F}L^1\big(\mathbb{R}^n\big).$$

Let $N \in \mathbb{N}$, and let $P(x) = \sum_{|\eta|\leq N} a_\eta x^\eta$, $x \in \mathbb{R}^n$, be a complex polynomial. Then we define

$$P(A) := \sum_{|\eta|\leq N} a_\eta A^\eta \text{ and } E_0 := \big\{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n),\ x \in E\big\}.$$

We know that the operator $P(A)$ is closable and that the following holds:

($\triangleright$) $\overline{E_0} = E$, $E_0 \subseteq \bigcap_{\eta\in\mathbb{N}_0^n} D(A^\eta)$, $\overline{P(A)_{|E_0}} = \overline{P(A)}$ and $\phi(A)P(A) \subseteq P(A)\phi(A) = (\phi P)(A)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Assuming that E is a function space on which translations are uniformly bounded and strongly continuous, the obvious choice for A_j is $-i\partial/\partial x_j$ (notice also that E can be consisted of functions defined on some bounded domain). If $P(x) = \sum_{|\eta|\leq N} a_\eta x^\eta$, $x \in \mathbb{R}^n$, and E is such a space (for example, $L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$), then it is not difficult to prove that $\overline{P(A)}$ is nothing else but the operator $\sum_{|\eta|\leq N} a_\eta(-i)^{|\eta|}\partial^{|\eta|}/\partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n} \equiv \sum_{|\eta|\leq N} a_\eta D^\eta$, acting with its maximal distributional domain. Recall that $P(x)$ is called r-coercive ($0 < r \leq N$) if there exist $M,\ L > 0$ such that $|P(x)| \geq M|x|^r$, $|x| \geq L$; by a corollary of the Seidenberg — Tarski theorem, the equality $\lim_{|x|\to\infty} |P(x)| = \infty$ implies in particular that $P(x)$ is r-coercive for some $r \in (0, N]$ (cf. [16, Remark 8.2.7]). For further information concerning the functional calculus for commuting generators of C_0-groups, see [4; 17; 18].

2. Formulation and proof of main results

In this section, we continue our previous research and prove some important results concerning the existence and uniqueness of mild solutions of the following semilinear degenerate relaxation equation

$$(\mathrm{DFP})_{sl}: \begin{cases} \mathbf{D}_t^{\alpha}\overline{P_2(A)}u(t) = \overline{P_1(A)}u(t) + f(t,u(t)), \quad t \geq 0, \\ u(0) = x, \end{cases}$$

where $0 < \alpha < 1$, the function $f(\cdot,\cdot)$ satisfies certain properties, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $iA_j,\ 1 \leq j \leq n$, are commuting generators of bounded C_0-groups on a Banach space E, and $A = (A_1, \cdots, A_n)$. If $f(t,u(t)) \equiv 0$, then we consider the associated Volterra integral equation

$$\overline{P_2(A)}u(t) = \overline{P_2(A)}x + \big(g_\alpha * \overline{P_1(A)}u\big)(t), \quad t \geq 0, \tag{2}$$

along with the problem $(\mathrm{DFP})_{sl}$. We need the following definition.

Definition 2. Let $0 < \alpha < 1$, let $C \in L(E)$ be injective, and let $C^{-1}\overline{P_1(A)}C = \overline{P_1(A)}$, $C^{-1}\overline{P_2(A)}C = \overline{P_2(A)}$. A strongly continuous operator family $(P_\alpha(t))_{t>0} \subseteq L(E)$ is said to be an $(\alpha, \alpha, \overline{P_1(A)}, \overline{P_2(A)}, C)$-resolvent family iff there exist $M \geq 1$ and $\omega \geq 0$ such that the mapping $t \mapsto \|t^{1-\alpha}P_\alpha(t)\|$, $t \in (0,1]$, is bounded, $\|P_\alpha(t)\| \leq Me^{\omega t}$, $t \geq 1$, $\lambda^\alpha \in \rho_C(\overline{P_2(A)}, \overline{P_1(A)})$ for $\Re\lambda > \omega$, and

$$\big(\lambda^\alpha\overline{P_2(A)} - \overline{P_1(A)}\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}P_\alpha(t)x\,dt, \quad \Re\lambda > \omega,\ x \in E.$$

The following theorem is backbone of this paper.

Theorem 1. *Suppose $0 < \alpha < 1$, $\omega \geq 0$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = \mathrm{dg}(P_1(x))$ and $N_2 = \mathrm{dg}(P_2(x))$.*

(i) *Let $N \in \mathbb{N}$, $r \in (0,N]$, let $Q(x)$ be an r-coercive complex polynomial of degree N, $a \in \mathbb{C} \setminus Q(\mathbb{R}^n)$, $\gamma > \frac{n\max(N, \frac{N_1+N_2}{\alpha})}{2r}$ (resp. $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}|\max(N, \frac{N_1+N_2}{\alpha})$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $\gamma' > \frac{\max(N, \frac{N_1+N_2}{\alpha})}{r}\frac{n}{2} + \frac{N_1+N_2}{r\alpha}(1-\alpha)$ ($\gamma' = n|\frac{1}{p} - \frac{1}{2}|\frac{\max(N, \frac{N_1+N_2}{\alpha})}{r} + \frac{N_1+N_2}{r\alpha}(1-\alpha)$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, and let*

$$\sup_{x\in\mathbb{R}^n} \Re\left(\left(\frac{P_1(x)}{P_2(x)}\right)^{1/\alpha}\right) \leq \omega. \tag{3}$$

Set

$$R_{\alpha,\gamma}(t) := \left(E_\alpha\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(a - Q(x)\big)^{-\gamma}\right)(A),\ t \geq 0,$$

$C := R_{\alpha,\gamma'}(0)$ and

$$P_{\alpha,\gamma'}(t) := t^{\alpha-1}\left(P_2(x)^{-1}E_{\alpha,\alpha}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(a - Q(x)\big)^{-\gamma'}\right)(A), \quad t > 0.$$

Then $(R_{\alpha,\gamma}(t))_{t\geq0} \subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_{\alpha,\gamma}(0))$-regularized resolvent family for (2), $(P_{\alpha,\gamma'}(t))_{t>0} \subseteq L(E)$ is a global

$(\alpha,\alpha,\overline{P_1(A)},\overline{P_2(A)},C)$-resolvent family, $(R_{\alpha,\gamma}(t))_{t\geq 0}$ is norm continuous provided $\gamma > \frac{n\max(N,\frac{N_1+N_2}{\alpha})}{2r}$, $(P_{\alpha,\gamma'}(t))_{t>0}$ is norm continuous provided $\gamma' > \frac{\max(N,\frac{N_1+N_2}{\alpha})}{r}\frac{n}{2}+\frac{N_1+N_2}{r\alpha}(1-\alpha)$,

$$\|R_{\alpha,\gamma}(t)\| \leq M(1+t^{n/2})e^{\omega t},\ t\geq 0,\ \textit{resp.},$$
$$\|R_{\alpha,\gamma}(t)\| \leq M\big(1+t^{n|\frac{1}{p}-\frac{1}{2}|}\big)e^{\omega t},\ t\geq 0, \tag{4}$$

and

$$\|P_{\alpha,\gamma'}(t)\| \leq Mt^{\alpha-1}\big(1+t^{1-\alpha+\frac{n}{2}}\big)e^{\omega t},\ t>0,\ \textit{resp.},$$
$$\|P_{\alpha,\gamma'}(t)\| \leq Mt^{\alpha-1}\big(1+t^{1-\alpha+n|\frac{1}{p}-\frac{1}{2}|}\big)e^{\omega t},\ t>0. \tag{5}$$

(ii) *Suppose $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\alpha}$ (resp. $\beta \geq n|\frac{1}{p}-\frac{1}{2}|\frac{(N_1+N_2)}{\alpha}$, if $E=L^p(\mathbb{R}^n)$ for some $1<p<\infty$), $\beta' > (1-\alpha+\frac{n}{2})\frac{(N_1+N_2)}{\alpha}$ (resp. $\beta' \geq (1-\alpha+n|\frac{1}{p}-\frac{1}{2}|)\frac{(N_1+N_2)}{\alpha}$, if $E=L^p(\mathbb{R}^n)$ for some $1<p<\infty$), $P_2(x)\neq 0$, $x\in\mathbb{R}^n$, and (3) holds. Set*

$$R_{\alpha,\beta}(t) := \left(E_\alpha\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(1+|x|^2\big)^{-\beta/2}\right)(A),\ t\geq 0,$$

$C := R_{\alpha,\beta'}(0)$, and

$$P_{\alpha,\beta'}(t) := t^{\alpha-1}\left(P_2(x)^{-1}E_{\alpha,\alpha}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(1+|x|^2\big)^{-\beta'/2}\right)(A),\quad t>0.$$

Then $(R_{\alpha,\beta}(t))_{t\geq 0} \subseteq L(E)$ is a global exponentially bounded $(g_\alpha,R_{\alpha,\beta}(0))$-regularized resolvent family for (2), $(P_{\alpha,\beta'}(t))_{t>0} \subseteq L(E)$ is a global $(\alpha,\alpha,\overline{P_1(A)},\overline{P_2(A)},C)$-resolvent family, $(R_{\alpha,\beta}(t))_{t\geq 0}$ is norm continuous provided $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\alpha}$, $(P_{\alpha,\beta'}(t))_{t>0}$ is norm continuous provided $\beta' > (1-\alpha+\frac{n}{2})\frac{(N_1+N_2)}{\alpha}$, (4) holds with $(R_{\alpha,\gamma}(t))_{t\geq 0}$ replaced by $(R_{\alpha,\beta}(t))_{t\geq 0}$ therein, and (5) holds with $(P_{\alpha,\gamma'}(t))_{t\geq 0}$ replaced by $(P_{\alpha,\beta'}(t))_{t\geq 0}$ therein.

Proof. Recall that $k=1+\lfloor n/2\rfloor$. In either choice of the regularizing operator C, we have $C^{-1}\overline{P_1(A)}C=\overline{P_1(A)}$ and $C^{-1}\overline{P_2(A)}C=\overline{P_2(A)}$ [19]. Furthermore, for every $j\in\mathbb{N}$, there exist uniquely determined real numbers $c_{l,j,\alpha}$ $(1\leq l\leq j)$ such that $E'_\alpha(z)=\alpha^{-1}E_{\alpha,\alpha}(z)$, $z\in\mathbb{C}$, as well as that $E_\alpha^{(j)}(z)=\sum_{l=1}^{j}c_{l,j,\alpha}E_{\alpha,\alpha j-(j-l)}(z)$, $z\in\mathbb{C}$. Using these facts, we have that:

1. For every multi-index $\eta\in\mathbb{N}_0^n$ with $|\eta|>0$, there exists $c_\eta>0$ such that
$$\left|D^\eta\left(\frac{P_1(x)}{P_2(x)}\right)\right| \leq c_\eta\big(1+|x|\big)^{|\eta|(N_1+N_2-1)},\quad x\in\mathbb{R}^n.$$
2. For every multi-index $\eta\in\mathbb{N}_0^n$ with $|\eta|>0$, for every $t\geq 0$ and for every $x\in\mathbb{R}^n$, we have:
$$D^\eta E_{\alpha,\alpha}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right) = \sum_{j=1}^{|\eta|}t^{\alpha j}E_{\alpha,\alpha}^{(j)}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)R_{\eta,j}(x) \tag{6}$$
$$= \sum_{j=1}^{|\eta|}t^{\alpha j}\sum_{l=1}^{j+1}\alpha c_{l,j+1,\alpha}E_{\alpha,\alpha j-j+l+\alpha-1}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)R_{\eta,j}(x), \tag{7}$$

where $R_{\eta,j}(x)$ can be represented as a finite sum of terms like $\prod_{q=1}^{s_j}D^{\eta_{j,q}}(\frac{P_1(x)}{P_2(x)})$ with $|\eta_{j,q}|>0$ $(1\leq q\leq s_j)$ and $|\eta_{j,1}|+\cdots+|\eta_{j,s_j}|\leq|\eta|$.

In the remaining part of proof, by M we denote a generic constant whose value may change from line to line. We get that

$$\big|R_{\eta,j}(x)\big| \leq M\big(1+|x|\big)^{|\eta|(N_1+N_2-1)}, \text{ provided } 1 \leq j \leq |\eta| \leq k \text{ and } x \in \mathbb{R}^n. \quad (8)$$

Arguing as in the proof of [20, Theorem 2.1], we can prove that, for every $t \geq 0$, $x \in \mathbb{R}^n$, and for every $j,\ l \in \mathbb{N}$ such that $1 \leq j \leq k$ and $1 \leq l \leq j+1$, the following holds:

$$\Big|E_{\alpha,\alpha j-(j-l)+\alpha-1}\big(t^\alpha P_1(x)/P_2(x)\big)\Big| \leq \\ \leq M\left[1+t^{1-(\alpha j-(j-l)+\alpha-1)}\big|P_1(x)/P_2(x)\big|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}e^{\omega t}\right]. \quad (9)$$

If $t \geq 0$ and $x \in \mathbb{R}^n$ satisfies $|t^\alpha P_1(x)/P_2(x)| \leq 1$, then the equation (6) yields:

$$\Big|D^\eta\Big(E_{\alpha,\alpha}\big(t^\alpha P_1(x)/P_2(x)\big)\Big)\Big| \leq M\Big(t^\alpha+t^{\alpha|\eta|}\Big)\big(1+|x|\big)^{|\eta|(N_1+N_2-1)}, \ |\eta| \leq k. \quad (10)$$

Suppose now that $1 \leq l \leq j+1$, $1 \leq j \leq |\eta| \leq k$, $t \geq 0$, $x \in \mathbb{R}^n$ and $|t^\alpha P_1(x)/P_2(x)| \geq 1$. Then it can be easily seen that the supposition $1-(\alpha j-(j-l)+\alpha-1) \geq 0$ implies

$$\big(N_1+N_2\big)\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}+|\eta|\big(N_1+N_2-1\big) \leq \\ \leq |\eta|\left(\frac{N_1+N_2}{\alpha}-1\right)+\big(N_1+N_2\big)\frac{1-\alpha}{\alpha}.$$

Using this estimate and (8), it readily follows that

$$\big|t^\alpha P_1(x)/P_2(x)\big|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}\big(1+|x|\big)^{|\eta|(N_1+N_2-1)} \leq \\ \leq Mt^{1-(\alpha j-(j-l)+\alpha-1)}\big(1+|x|\big)^{|\eta|\left(\frac{N_1+N_2}{\alpha}-1\right)+(N_1+N_2)\frac{1-\alpha}{\alpha}},$$

provided $1-(\alpha j-(j-l)+\alpha-1) \geq 0$. On the other hand, it is clear that

$$\big|t^\alpha P_1(x)/P_2(x)\big|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}\big(1+|x|\big)^{|\eta|(N_1+N_2-1)} \leq M\big(1+|x|\big)^{|\eta|(N_1+N_2-1)}, \quad (11)$$

provided $1-(\alpha j-(j-l)+\alpha-1) \leq 0$. Then, for every $t \geq 0$, $x \in \mathbb{R}^n$ and $0 < |\eta| \leq k$, the following holds (cf. (7), (9)–(11) and [19, (2.6), (2.7)]):

$$\left|D^\eta E_{\alpha,\alpha}\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\right| \leq M\big(1+t^{1-\alpha}\big)\big(1+t^{|\eta|}\big)e^{\omega t}\big(1+|x|\big)^{|\eta|\left(\frac{N_1+N_2}{\alpha}-1\right)+(N_1+N_2)\frac{1-\alpha}{\alpha}}; \quad (12)$$

observe that the inequality $\Re(t^\alpha(P_1(x)/P_2(x))^{1/\alpha}) \leq \omega t$, $t \geq 0$, $x \in \mathbb{R}^n$, and Lemma 1 together imply that the previous estimate also holds in the case that $|\eta| = 0$. Define

$$G_{\alpha,\gamma'}(t) := \left(P_2(x)^{-1}E_\alpha\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(a-Q(x)\big)^{-\gamma'}\right)(A), \quad t \geq 0,$$

in the case of consideration of (i), resp.

$$G_{\alpha,\gamma'}(t) := \left(P_2(x)^{-1}E_\alpha\left(t^\alpha\frac{P_1(x)}{P_2(x)}\right)\big(1+|x|^2\big)^{-\beta'/2}\right)(A), \quad t \geq 0,$$

in the case of consideration of (ii). Then the identity

$$E_\alpha\big(t^\alpha P_1(x)/P_2(x)\big) = \int_0^t g_{1-\alpha}(t-s)s^{\alpha-1}E_{\alpha,\alpha}\big(s^\alpha P_1(x)/P_2(x)\big)\,ds, \quad t>0,\ x\in\mathbb{R}^n,$$

(cf. [12, p. 212, l.4]), shows that

$$G_{\alpha,\gamma'}(t) = \big(g_{1-\alpha} * P_{\alpha,\gamma'}\big)(t),\ t>0, \quad \text{and } G_{\alpha,\beta'}(t) = \big(g_{1-\alpha} * P_{\alpha,\beta'}\big)(t),\ t>0.$$

The proof can be completed routinely by using the estimate (12), the above equalities and the argumentation used in the proofs of [19, Theorem 2.1]. □

It is worth noting that the assertion of Theorem 1 continues to hold, with appropriate technical modifications, in the case that $E = C_b(\mathbb{R}^n)$ or $E = L^\infty(\mathbb{R}^n)$, and that the additional refinement of lower bounds for the numbers $\gamma,\ \gamma',\ \beta,\ \beta'$ can be proved. Suppose now that $T>0$, the requirements of Theorem 1 (i) or Theorem 1 (ii) hold, $C := R_{\alpha,\gamma'}(0)$ ($C := R_{\alpha,\beta'}(0)$) in the case of consideration of Theorem 1 (i) (Theorem 1 (ii)) and $x \in R(C)$. Following the analysis of non-degenerate case carried out in [12] and [13] (another way to see that the subsequent definition of a mild solution of the problem $(\mathrm{DFP})_{sl}$ is correct in the case that $f(t,u(t)) = f(t)$ satisfies (P1) is to take the Laplace transform of both sides of the equality $\mathbf{D}_t^\alpha \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t) + f(t)$ by making use of the formula [1, (1.23)], on the one hand, and to compute the Laplace transform of the right hand side of equality (13) below, on the other hand), it will be said that a continuous function $t \mapsto u(t)$, $t\in[0,T]$ is a mild solution of the semilinear abstract degenerate Cauchy problem $(\mathrm{DFP})_{sl}$ on $[0,T]$ iff the mapping $t \mapsto C^{-1}f(t,u(t))$, $t\in[0,T]$, is well-defined and continuous, as well as

$$u(t) = R_\alpha(t)C^{-1}x + \int_0^t P_\alpha(t-s)C^{-1}f(s,u(s))\,ds, \quad t\in[0,T]. \tag{13}$$

Define the operator $Q_\alpha : C([0,T]:E) \to C([0,T]:E)$ by

$$\big(Q_\alpha u\big)(t) := R_\alpha(t)C^{-1}x + \int_0^t P_\alpha(t-s)C^{-1}f(s,u(s))\,ds, \quad t\in[0,T].$$

The most common technique to proving existence and uniqueness of mild solutions of semilinear fractional evolution equations is to apply some of the fixed point theorems; in our concrete situation, we must prove that the mapping $Q_\alpha(\cdot)$ has a unique fixed point. Not aspiring completeness of analysis here, we shall only state and prove the following adaptation of [21, p. 184, Theorem 1.2] to close the whole paper.

Theorem 2. *Let $T>0$, let $x\in R(C)$, and let the requirements of Theorem* 1 (i) *or Theorem* 1 (ii) *hold. Put $C := R_{\alpha,\gamma'}(0)$, in the case of Theorem* 1 (i), *and $C := R_{\alpha,\beta'}(0)$, in the case of Theorem* 1 (ii). *Suppose that the mapping $C^{-1}f : [0,T]\times E \to E$ is continuous in t on $[0,T]$ and uniformly Lipschitz continuous (with constant L) on E. Then the semilinear fractional Cauchy problem $(DFP)_{sl}$ has a unique mild solution $u \in C([0,T]:E)$. Moreover, the mapping $x \to u(\cdot)$ is Lipschitz continuous from $[R(C)]$ into $C([0,T]:E)$.*

Proof. Set $M := \max_{t\in(0,T]}(t^{1-\alpha}\Gamma(\alpha)\|P_\alpha(t)\|)$. Arguing as in the proof of [21, p. 184, Theorem 1.2], we get that, for every $u,\ v\in C([0,T]:E)$,

$$\Big\|\big(Q_\alpha^n u\big)(t) - \big(Q_\alpha^n v\big)(t)\Big\|_{C([0,T]:E)} \le \frac{\big(MLT^\alpha\big)^n}{\Gamma(n\alpha+1)}\|u-v\|_{C([0,T]:E)},\ n\in\mathbb{N},\ t\in[0,T].$$

For a sufficiently large number $n \in \mathbb{N}$, one has

$$\frac{(MLT^{\alpha})^n}{\Gamma(n\alpha+1)} < 1,$$

so that a well known extension of the Banach contraction principle implies that the mapping $Q_{\alpha}(\cdot)$ has a unique fixed point, finishing the proof of existence and uniqueness of mild solutions of problem $(\mathrm{DFP})_{sl}$ on $[0,T]$. Keeping in mind a Gronwall-type inequality [2, Lemma 6.19, p. 111], the remaining part of proof follows similarly as in that of [21, p. 184, Theorem 1.2]. □

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О ПОЛУЛИНЕЙНЫХ ВЫРОЖДЕННЫХ РЕЛАКСАЦИОННЫХ УРАВНЕНИЯХ, СВЯЗАННЫХ С АБСТРАКТНЫМИ ДИФФЕРЕНЦИАЛЬНЫМИ ОПЕРАТОРАМИ

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Основная цель данной работы — доказать некоторые новые результаты о полулинейных вырожденных релаксационных уравнениях, связанных с абстрактными дифференциальными операторами. Для того чтобы это сделать, использована техника мультипликаторов Фурье.

Ключевые слова: *абстрактное полулинейное вырожденное релаксационное дифференциальное уравнение, функция Миттаг-Лёффлера, дробная производная Капуто, $(\alpha, \alpha, \overline{P_1(A)}, \overline{P_2(A)}, C)$-разрешающее семейство, абстрактный дифференциальный оператор.*

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