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GENERALIZATIONS OF THE KAUFFMAN POLYNOMIAL FOR KNOTS IN THE THICKENED SURFACE OF GENUS 2

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In this paper, we discuss two invariants, which are results of different approaches to the generalization of the Kauffman polynomial for the case of knots in the thickened surface of genus 2. The first generalization proposes to distinguish only two types of curves (cut and noncut) and seems to be rather strong to prove non equivalence of a big number of knots in the thickened surface of genus 2. However, the first generalization provides no tools to determine if a knot can be realised in the thickened surface having a smaller genus. In its turn, the second generalization proposes in addition to distinguish isotopy types of noncut curves. As a result, such an invariant is enough to perform the destabilisation, if it is possible, by the following steps. First, determine a destabilisation curve or prove that such a curve does not exist. Second, if a destabilisation curve exists, use a sequence of elementary transformations to reduce the found destabilisation curve to the standard form. Finally, reduce isotopy types of noncut curves involved in the terms of the generalized Kauffman polynomial to obtain the generalized Kauffman polynomial of the same knot realised in the thickened surface having a smaller genus. Computational examples show the effectiveness of the second generalization and the weakness of the first generalization in the destabilisation.

Keywords: *Kauffman polynomial, knot, thickened surface of genus 2, isotopy types of curves.*

Introduction

One of the main problems of the knot theory is to distinguish the objects under study. This approach involves the problem to construct and compute knot invariants and see if some of them are helpful in the considered particular situation. During more than 30 years, the Kauffman polynomial of classical knots [1] (see also [2] for the original version called the Jones polynomial) and its generalizations for the case of knots in other 3-dimensional manifolds [3–8] remain among the best invariants. For example, the generalized Kauffman polynomial turned out to be enough to prove non equivalence of all prime knots [3] in the thickened torus having diagrams with at most 5 crossings.

In this paper, we discuss two invariants, which are results of different approaches to the generalization of the Kauffman polynomial for the case of knots in the thickened surface of genus 2. In contrast to the usual case of classical knots, the generalized versions take into account types of curves.

The first generalization proposes to distinguish only two types of curves (cut and noncut) and seems to be rather strong to proof non equivalence of a big number of knots in the thickened surface of genus 2. However, the first generalization provides no tools to determine if a knot can be realised in the thickened surface having a smaller genus. Indeed, the obtained generalized Kauffman polynomials of knots in the thickened surface

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of genus 2 are not equivalent to the generalized Kauffman polynomial of the same knots in the thickened surfaces having a smaller genus.

In its turn, the second generalization proposes in addition to distinguish isotopy types of noncut curves. As a result, such an invariant is enough to perform a destabilisation, if it is possible, during the following steps. First, to determine if a knot in the thickened surface of genus 2 can be realised in the thickened surface having a smaller genus. To this end, determine a destabilisation curve or prove that such a curve does not exist. Second, if the destabilisation is possible, use a sequence of elementary transformations to reduce the found destabilisation curve to the standard form. Finally, reduce isotopy types of noncut curves involved in the terms of the generalized Kauffman polynomial to obtain the generalized Kauffman polynomial of the same knot realised in the thickened surface having a smaller genus.

The paper is organized as follows. Section 1 gives the necessary definitions of knots and manifolds. Section 2 describes some generalizations of the Kauffman polynomial for knots in the thickened torus and in the thickened surface of genus 2. Section 3 considers the generalized Kauffman polynomial taking into account isotopy types of curves as a tool to determine possibility to realize a knot in the thickened surface of genus 2 as a knot in the thickened surface having a smaller genus. Computational examples presented in Section 4 show implementation of the steps to determine and perform the destabilisation, i. e. the effectiveness of the generalization taking into account isotopy types of noncut curves and the weakness of the generalization identified all noncut curves in the question on the determination of the destabilisation.

1. Definitions

A direct product of two copies of an 1-dimensional sphere S^1 is called a 2-dimensional torus $T = S^1 \times S^1$. Further, for shortness, we refer to a 2-dimensional torus T as a torus T. Fig. 1 (a) shows an example of a torus T endowed with an oriented pair «meridian-longitude» of T.

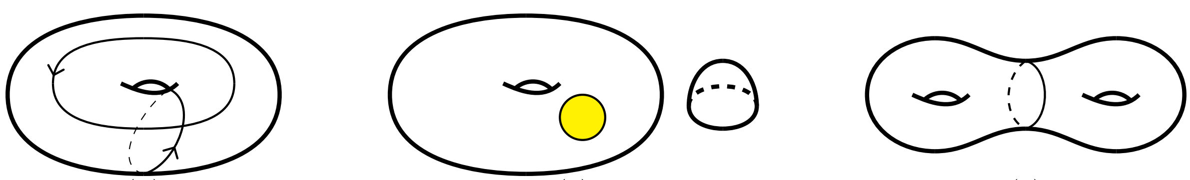

Fig. 1. (a) A torus T endowed with an oriented pair «meridian-longitude», (b) a torus T^o with a hole and a disk D, (c) a surface T_2 formed by gluing of two copies of a torus T^o with a hole

A surface F^o with a hole is obtained from the original surface F by removing the interior of a 2-dimensional disk D^2. Further, for shortness, we refer to a 2-dimensional disk D^2 as a disk D. Fig. 1 (b) shows an example: a torus T^o with a hole is obtained from a torus T by removing the interior of a disk D. Hereinafter, we write ^{o} to show that a surface has one hole, ^{oo} to show that a surface has two holes.

In the torus T, we distinguish two types of curves: cut and noncut. A simple closed curve $C \subset T$ is said to be cut, if the complement $T \backslash C$ is formed by a disk D and a torus T^o with a hole. A simple closed curve $C \subset T$ is said to be noncut, if the complement $T \backslash C$ consists of the unique component, which is an annulus $\mathcal{A}$, i. e. a 2-dimensional sphere S^{oo} with two holes. For example, both curves given in Fig. 1 (a) are noncut, while a curve shown in Fig. 1 (b) is cut.

In their turn, noncut curves can be distinguished according to their isotopy types. For a fixed oriented pair «meridian-longitude» of the torus T (throughout the paper, see Fig. 1 (a)), we say that an oriented noncut curve $C \subset T$ has the isotopy type (p, q), if C goes p times along the longitude and q times along the meridian of the torus T, where the number p (respectively, q) is positive, if the direction of orientation of C coincides with the direction of orientation of the longitude (respectively, the meridian). Since the direction of orientation of C is arbitrary, we consider isotopy types (p, q) and $(-p, -q)$ to be equal. It is well known that the greatest common divisor $gcd(p, q) = 1$, otherwise C has self-intersections. In practice, the number p (respectively, q) is calculated as an intersection number of the curve C and the meridian (respectively, the longitude) of the torus T. Recall that the intersection number of two oriented curves is defined as a sum of signs of their intersection points. Here we say that an intersection point has the sign «+1» if the rotation from the direction of the first curve to the direction of the second curve is counterclockwise. Otherwise the intersection point has the sign «−1».

By a 2-dimensional surface T_2 of genus 2 we mean a surface formed by gluing of two copies of a torus T^o with a hole constructed by identifying their holes, see Fig. 1 (c). Here each torus T^o is called a handle of a 2-dimensional surface T_2 of genus 2. Further, for shortness, we refer to a 2-dimensional surface T_2 of genus 2 as a surface T_2.

Let us define types of simple closed curves, which can be considered in a surface T_2.

A simple closed curve $C \subset T_2$ is said to be cut, if the complement $T_2 \backslash C$ consists of two components. In the surface T_2, a cut curve C can be either trivial, i. e. bounding a disk D, or nontrivial. In the first case, the complement $T_2 \backslash C$ is formed by a disk D and a surface ${T_2}^o$ with a hole. In the second case, the complement $T_2 \backslash C$ is formed by two copies of a torus T^o with a hole. A simple closed curve $C \subset T_2$ is said to be noncut, if the complement $T_2 \backslash C$ consists of the unique component. Namely, the complement $T_2 \backslash C$ is a torus T^{oo} with two holes. Two noncut simple closed curves $C_1,\ C_2 \subset T_2$ are said to be parallel to each other, if the complement $T_2 \backslash (C_1 \cup C_2)$ consists of two components, which are a torus T^{oo} with two holes and an annulus $\mathcal{A}$. Fig. 2 (a) shows examples: $C_1,\ C_2 \subset T_2$ are two noncut curves parallel to each other, while $C_3,\ C_4 \subset T_2$ are nontrivial and trivial cut curves, respectively.

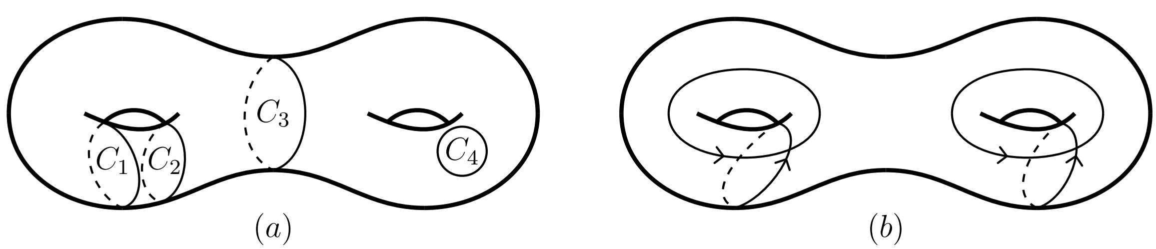

Fig. 2. (a) Examples of curves in the surface T_2, (b) a surface T_2 endowed with oriented pairs «meridian-longitude» of its handles

For two fixed oriented pairs «meridian-longitude» of handles of the surface T_2 (throughout the paper, see Fig. 2 (b)), we say that an oriented noncut curve $C \subset T_2$ has the isotopy type (a, b, c, d), if C goes a times along the longitude and b times along the meridian of the left handle of the surface T_2, and c times along the longitude and d times along the meridian of the right handle of the surface T_2. As well as in the case of the torus T, the signs of the numbers a, b, c, d are positive, if the direction of orientation of C coincides with the direction of the corresponding longitude or meridian. Since the direction of orientation of C is arbitrary, we consider isotopy types (a, b, c, d) and $(-a, -b, -c, -d)$ to be equal. In contrast to the case of the torus T, there exist noncut curves without self-intersections such that $gcd(a, b) \neq 1$ or $gcd(c, d) \neq 1$, see Fig. 6 (the state $BABB$) for the curve having the isotopy type $(2, 1, 0, -2)$. In practice, the numbers a and c (respectively, b and d) are calculated as intersection numbers of the curve C and the corresponding meridian (respectively, longitude) of the surface T_2.

Lemma 1. *In the surface T_2, the intersection number of two noncut curves having isotopy types (a, b, c, d) and (a', b', c', d') equals to $(a \cdot b' - b \cdot a') + (c \cdot d' - d \cdot c')$.*

Proof. For the torus T, it is well known that the intersection number of two noncut curves having isotopy types (p, q) and (p', q') equals to $(p \cdot q' - q \cdot p')$. For the surface T_2, take into account that pairs «meridian-longitude» do not intersect each other. □

Consider a surface T_2 and an interval $I = [0, 1]$. By a thickened surface of genus 2 we mean a 3-dimensional manifold homeomorphic to the direct product $T_2 \times I$. A thickened torus $T \times I$ is defined by analogy.

A smooth embedding of m closed curves in the $\mathrm{Int}(T_2 \times I)$ is called a link in $T_2 \times I$ having m components and denoted by $L \subset T_2 \times I$. An one-component link in $T_2 \times I$ is called a knot in $T_2 \times I$ and denoted by $K \subset T_2 \times I$. Knots and links in the thickened torus $T \times I$ are defined by analogy. Hereinafter, we consider knots under assumption that the corresponding constructions for links are the same with the exclusion of some cases discussed individually (e. g., the writhe).

As in the classical case, knots in $T_2 \times I$ can be given by their diagrams. A diagram $D \subset T_2$ of a knot $K \subset T_2 \times I$ is defined by analogy with the diagram of the classical knot except that the knot is projected into the surface T_2 instead of the plane.

Let $D \subset T_2$ be a knot diagram. We say that a noncut curve $C \subset T_2 \times I$ is a destabilisation curve for the pair (D, T_2), if an intersection of C and D is empty. In order to perform a destabilization of the surface T_2, it is enough to cut T_2 along a destabilisation curve C and glue each component of the boundary by a disk D. Fig. 3 shows a torus T as a result of a destabilization of the surface T_2 of genus 2.

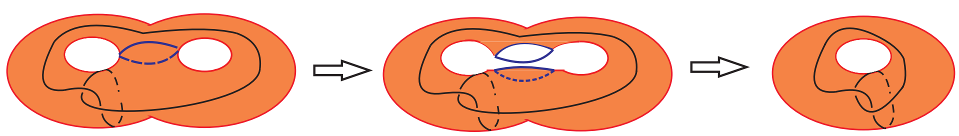

Fig. 3. A destabilization of the surface of genus 2

2. Generalizations of the Kauffman Polynomial

Consider generalizations of the Kauffman polynomial for the cases of the torus T and the surface T_2.

Let us recall a definition of the Kauffman polynomial $\widetilde{\mathcal{X}}(\cdot)$ of two variables a and x [6]. Note that such an invariant is used in [5] to construct the table of knots in the solid torus (the thickened annulus $\mathcal{A} \times I$).

Let D be a diagram of a knot in the thickened torus $T \times I$. Endow each angle of each crossing of D with a marker A or B according to the rule given in the center of Fig. 4 (a). Each state s of the diagram D is defined by a combination of ways to smooth each crossing of D such as to join together either two angles endowed with a marker A, or two angles endowed with a marker B, see Fig. 4 (a) on the left and right, respectively. Obviously, if the diagram D has n crossings, then there exist exactly 2^n states of D.

By the writhe of an oriented knot diagram D with n crossings we mean the sum over all crossings of D, i. e. $w(D) = \sum_{i=1}^{n} \varepsilon(i)$, where $\varepsilon(i)$ is a sign of the i-th crossing of D defined by the rules given in Fig. 4 (b). Note that the writhe of an oriented link

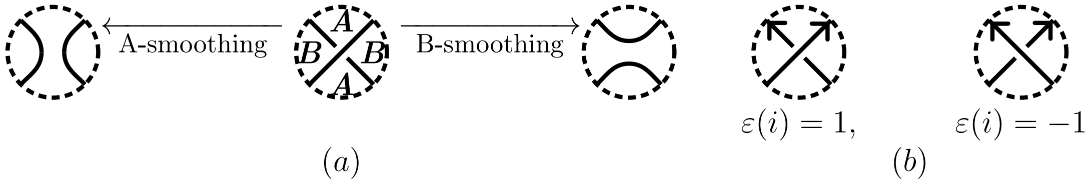

Fig. 4. (a) A- and B-smoothings of a crossing, (b) rules to define the sign $\varepsilon(i)$ of the i-th crossing

diagram is the sum of signs of only those crossings of D that are self-intersections of the components.

The exact formula of the generalized Kauffman polynomial $\widetilde{\mathcal{X}}(\cdot)$ [6] is as follows:

$$\widetilde{\mathcal{X}}(D) = (-a)^{-3w(D)} \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)} x^{\delta(s)}. \tag{1}$$

Here $\alpha(s)$ and $\beta(s)$ are the numbers of markers A and B in the given state s, while $\gamma(s)$, $\delta(s)$ are the numbers of cut and noncut curves in the torus obtained by smoothing of all crossings according to the state s, and $w(D)$ is the writhe of D. The sum is taken over all 2^n states of D.

Let us define a stronger invariant that takes into account the isotopy type (p, q) of noncut curves and was proposed by S.V. Matveev. In contrast to $\widetilde{\mathcal{X}}(\cdot)$ (1), the variable x is endowed with two numbers p and q to show the isotopy type (p, q) of the corresponding noncut curve:

$$\widetilde{\mathcal{X}}(D)_{pq} = (-a)^{-3w(D)} \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)} x_{p,q}^{\delta(s)}. \tag{2}$$

We say that two generalized Kauffman polynomials $\widetilde{\mathcal{X}}(\cdot)_{pq}$ and $\widetilde{\mathcal{X}}(\cdot)'_{pq}$ are equivalent to each other, if there exists a homeomorphism of the torus T onto itself that takes the set of isotopy type (p_i, q_i) noncut curves corresponding to the terms of $\widetilde{\mathcal{X}}(\cdot)_{pq}$ to the set of isotopy type (p'_i, q'_i) noncut curves corresponding to the terms of $\widetilde{\mathcal{X}}(\cdot)'_{pq}$ under the same coefficients of the polynomials $\widetilde{\mathcal{X}}(\cdot)_{pq}$ and $\widetilde{\mathcal{X}}(\cdot)'_{pq}$ for all i.

Let us construct two generalizations (for both (1) and (2)) in the case of the surface T_2.

In order to construct the simplest generalization for the case of the surface T_2, we propose to distinguish only two types of curves: cut (including both trivial and nontrivial) and noncut (including both parallel to each other and not). The resulting formula of $\widehat{\mathcal{X}}(\cdot)$ coincides with $\widetilde{\mathcal{X}}(\cdot)$ (1) except in the curves belong to T_2. Such a generalization is enough stronger to distinct two knots in $T_2 \times I$, but can not determine if a knot in $T_2 \times I$ can be realized in $T \times I$ or not. Moreover, the same knot can have different polynomials $\widetilde{\mathcal{X}}(\cdot)$ and $\widehat{\mathcal{X}}(\cdot)$, see an example in Section 4.

Therefore, we propose a stronger generalization taking into account the isotopy type (a, b, c, d) of noncut curves. In Section 3, we consider such a generalization as a tool to determine possibility to realize a knot in the thickened surface of genus 2 as a knot in the thickened surface having smaller genus. In this case, the exact formula of the generalized Kauffman polynomial is as follows:

$$\widehat{\mathcal{X}}(D)_{abcd} = (-a)^{-3w(D)} \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)} \cdot \prod_{a,b,c,d:\ a^2+b^2+c^2+d^2 \neq 0} x_{a,b,c,d}^{\delta_{a,b,c,d}(s)}. \tag{3}$$

In contrast to (1), $\delta_{a,b,c,d}(s)$ is the number of noncut curves having the isotopy type (a, b, c, d) in the surface T_2, and the variable x is endowed with four numbers to show the isotopy type of the noncut curve.

We say that two generalized Kauffman polynomials $\widehat{\mathscr{X}}(\cdot)_{abcd}$ and $\widehat{\mathscr{X}}(\cdot)'_{abcd}$ are equivalent to each other, if there exists a homeomorphism of the surface T_2 onto itself that takes the set of isotopy type (a_i, b_i, c_i, d_i) noncut curves involved in the terms of $\widehat{\mathscr{X}}(\cdot)_{abcd}$ to the set of isotopy type (a'_i, b'_i, c'_i, d'_i) noncut curves involved in the terms of $\widehat{\mathscr{X}}(\cdot)'_{abcd}$ under the same coefficients of $\widehat{\mathscr{X}}(\cdot)_{abcd}$ and $\widehat{\mathscr{X}}(\cdot)'_{abcd}$ for all i.

Remark 1. For knots in the surface T_2, it is possible to propose more difficult generalizations that distinguish the types of cut curves. To this end, it is enough to use an additional variable d and to replace $(-a^2 - a^{-2})^{\gamma(s)}$ with $(-a^2 - a^{-2})^{\gamma_1(s)} d^{\gamma_2(s)}$, where $\gamma_1(s)$ and $\gamma_2(s)$ are the numbers of trivial and nontrivial cut curves, in both $\widehat{\mathscr{X}}(\cdot)$ (1) and $\widehat{\mathscr{X}}(\cdot)_{abcd}$ (3). Such generalizations can be useful in some more difficult situations that are not considered in the present paper.

Lemma 2. *The generalized Kauffman polynomials $\widehat{\mathscr{X}}(\cdot)$ and $\widehat{\mathscr{X}}(\cdot)_{abcd}$ are invariant under*

(i) *ambient isotopy,*

(ii) *simultaneous switching of all crossings,*

(iii) *homeomorphisms of the surface T_2 onto itself*

up to change of variables $a \to a^{-1}$ and one-to-one change of variables $x_{a,b,c,d}$ connected with taken of the set of isotopy types of curves involved in the first $\widehat{\mathscr{X}}(\cdot)_{abcd}$ to the set involved in the second $\widehat{\mathscr{X}}(\cdot)'_{abcd}$ under the same coefficients of the polynomials $\widehat{\mathscr{X}}(\cdot)_{abcd}$ and $\widehat{\mathscr{X}}(\cdot)'_{abcd}$.

Proof. Items (i) and (ii) are proved as well as in the classical case, see, for example, [1]. Item (iii) is obvious, since any homeomorphism remains numbers of cut and noncut curves the same and changes the isotopy type of any noncut curve by the same rule. □

3. $\widehat{\mathscr{X}}(\cdot)_{abcd}$ as a tool to determine possibility to realize $K \subset T_2 \times I$ as a knot in the thickened surface having smaller genus

Theorem 1. *The generalized Kauffman polynomial $\widehat{\mathscr{X}}(\cdot)_{abcd}$ is enough to determine a realization of a knot K in the thickened surface $T_2 \times I$ of genus 2 as a knot in the thickened torus $T \times I$ or in the thickened annulus $\mathcal{A} \times I$ (solid torus), if such a realization exists.*

Proof. Step 1. Lemma 3 determines if the given knot $K \subset T_2 \times I$ can be realized as a knot in $T \times I$.

Lemma 3. *Suppose that the generalized Kauffman polynomial $\widehat{\mathscr{X}}(K)_{abcd}$ of a knot $K \subset T_2 \times I$ contains terms corresponding to m noncut curves having nonequivalent isotopy types (a_i, b_i, c_i, d_i), $i \in \{1, 2, ..., m\}$. If there exists a destabilisation curve C for the pair (D, T_2), where $D \subset T_2$ is some diagram of the knot K, then the isotopy type (a, b, c, d) of C is a nonzero solution to the following system of m linear equations of 4 variables a, b, c, d with coefficients a_i, b_i, c_i, d_i, where $i \in \{1, 2, ..., m\}$:*

$$b_i \cdot a - a_i \cdot b + d_i \cdot c - c_i \cdot d = 0, \quad i \in \{1, 2, ..., m\}.$$

If the system has only zero solution, then the knot K can not be realized in $T \times I$.

Proof. The system is based on the following two ideas. First, the intersection number of two noncut curves can be calculated according to Lemma 1. Second, the intersection number of the desired noncut curve C and any of m noncut curves involved in the terms of $\widehat{\mathcal{X}}(K)_{abcd}$ should be equal to zero. □

Step 2. If Lemma 3 shows that the knot K can be realized in $T \times I$, then we can perform a destabilisation of the surface T_2 along the obtained destabilisation curve C.

Step 2.1. In order to make obvious the resulting isotopy types of all m noncut curves corresponding to the terms of $\widehat{\mathcal{X}}(K)_{abcd}$, we transform the isotopy type of the destabilisation curve C to one of the standard isotopy types described in Lemma 4. To this end, we use twists of handles defined as follows. In order to twist a handle along a closed noncut curve, it is enough to cut the handle along the curve, twist one of the obtained boundaries by 360 degrees, and glue the boundaries together.

Lemma 4. *For any noncut curve C having the isotopy type (a,b,c,d) in the surface T_2, there exists a sequence of twists of handles of the surface T_2 that reduces C to the noncut curve having one of the following standard isotopy types:*

(i) $(0,1,0,-1)$, *if* $a^2+b^2 \neq 0$ *and* $c^2+d^2 \neq 0$,

(ii) $(0,1,0,0)$, *if* $a^2+b^2 \neq 0$ *and* $c^2+d^2 = 0$,

(iii) $(0,0,0,1)$, *if* $a^2+b^2 = 0$ *and* $c^2+d^2 \neq 0$.

Proof. In Cases (ii) and (iii), use twists of the left or the right handle of the surface T_2, respectively. In Case (i), use twists of both handles. □

Note that all the noncut curves are involved in the same transformations as the destabilisation curve C. Therefore, in order to use Lemma 4 correctly, we construct the matrix $M_{(m+1)\times 4}$, which rows are formed by the isotopy types (a,b,c,d) and (a_i,b_i,c_i,d_i), $i \in \{1,2,...,m\}$, of the destabilisation curve C and m noncut curves corresponding to the terms of $\widehat{\mathcal{X}}(K)_{abcd}$. Then, we use the following elementary transformations of columns corresponding to twists of the left handle of the surface T_2: add the 1-st column multiplied by some number to the 2-nd column, and vice versa. Twists of the right handle of the surface T_2 involve similar transformations of the 3-rd and the 4-th columns.

Step 2.2. Now we can perform the destabilisation of the surface T_2 along the destabilisation curve C having one of the standard isotopy types described in Lemma 4. As a result, the curve C is reduced to a cut one in the torus T. Therefore, in $\widehat{\mathcal{X}}(K)_{abcd}$, we replace the corresponding terms of the form $x_{0,1,0,-1}$, $x_{0,1,0,0}$, or $x_{0,0,0,1}$ with $(-a^2-a^{-2})$, if such terms exist. In its turn, each noncut curve having isotopy type (a_i,b_i,c_i,d_i) in the surface T_2 is reduced to some noncut curve having the isotopy type (p_i,q_i), $i \in \{1,2,...,m\}$, in the torus T according to the rules defined by the destabilisation curve C in Lemma 5.

Lemma 5. *Let $C \subset T_2$ be a noncut curve having the isotopy type (a,b,c,d) and zero intersection number with a noncut curve C'. The destabilisation of the surface T_2 performed along C' reduces C to a noncut curve having isotopy type (p,q) in the torus T, where*

(i) $p=a=c$ *and* $q=b+d$, *if* C' *has the isotopy type* $(0,1,0,-1)$,

(ii) $p=c$ *and* $q=d$, *if* C' *has the isotopy type* $(0,1,0,0)$,

(iii) $p=a$ *and* $q=b$, *if* C' *has the isotopy type* $(0,0,0,1)$.

Proof. Statement (i) is obvious, since the destabilisation of the surface T_2 is performed along the same curve C' as in Fig. 3. Indeed, since the intersection number of C and C' is zero, then it follows from Lemma 1 that $a=c$, therefore, the intersection number with the meridian of the torus T equals to $p=a=c$. In its turn, the intersection number with the longitude of the torus T equals to $q = b + d$, since orientations of longitudes and orientations of meridians of handles of the surface T_2 coincide.

In Statement (ii), the curve C' can be considered as the curve C_1 shown in Fig. 2 (a), while zero intersection number leads to $a = 0$. For any b, after a destabilisation we can throw a thread over the glued disks, therefore, we obtain that $p = c$ and $q = d$. Statement (iii) is symmetrical to Statement (ii). □

Step 3. By analogy with Lemma 3, it is easy to see that if the obtained generalized Kauffman polynomial $\widetilde{\mathcal{X}}\,(K')_{pq}$ contains at least two variables of the form $x_{p,q}$ corresponding to noncut curves having different isotopy types, then there exists no destabilisation curve for any diagram of the knot $K' \subset T \times I$. Note that, in order to determine the knot $K' \subset T \times I$ corresponding to the original knot $K \subset T_2 \times I$, we can use the tables of knots in the thickened torus $T \times I$ and their invariants. For example, consider all terms of the form $x_{p,q}$ to be equal to x and see [6] for knots having diagrams with at most five crossings.

If all noncut curves have the same isotopy type (p, q), i. e. $p_i = p$ and $q_i = q$ for all $i \in \{1, 2, ..., m\}$, then we can apply the destabilisation of the torus T along the noncut curve having the isotopy type (p, q). Therefore, the knot $K' \subset T \times I$ can be realized as a knot K'' in the thickened annulus $\mathcal{A} \times I$, i. e. can be found in the table of knots in the solid torus, see [5] for knots having diagrams with at most six crossings. To this end, it is also necessary to consider all terms of the form $x_{p,q}$ to be equal to x.

This completes the proof of Theorem 1. □

4. Computational Examples

In order to illustrate an application of Theorem 1, we consider the generalized Kauffman polynomials $\widehat{\mathcal{X}}\,(\cdot)_{abcd}$ of three knots given in Fig. 5.

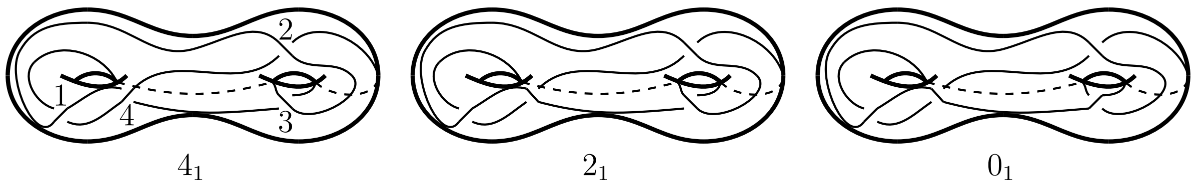

Fig. 5. Diagrams of knots in the thickened surface $T_2 \times I$

First of all, consider the diagram 4_1 shown in Fig. 5, where each crossing is endowed with an ordered number. Each state is given by the sequence of 4 letters, each of which can be either A or B depending on the considered type of smoothing given in Fig. 4 (a). Enumerate all 2^4 possible ways to smooth crossings. The result is presented in Fig. 6, where the record of the form (a, b, c, d) means that the given state involves a noncut curve having the isotopy type (a, b, c, d). Here $(0, 0, 0, 0)$ denotes a cut curve.

Therefore, $\widehat{\mathcal{X}}\,(4_1)_{abcd} = a^{-16}x_{0,0,0,1}x_{2,1,0,-1} + a^{-14}x_{0,0,0,1}x_{0,1,0,-1} - a^{-10}x_{0,0,0,1}x_{0,1,0,-1} - a^{-6}x_{0,0,1,0}x_{0,1,-1,0} + 2(a^{-12} - a^{-8})x_{0,1,0,0} + a^{-10}x_{0,0,1,0}x_{0,1,-1,0} + a^{-12}x_{0,1,-2,0} + a^{-14}x_{2,1,0,0} + a^{-14}x_{2,1,0,-2}$, while $\widehat{\mathcal{X}}\,(4_1) = (2a^{-14} + 3a^{-12} - 2a^{-8})x + (a^{-16} + a^{-14} - a^{-6})x^2$.

In order to calculate $\widehat{\mathcal{X}}\,(2_1)_{abcd}$ and $\widehat{\mathcal{X}}\,(0_1)_{abcd}$, we can use Fig. 6 as follows. For $\widehat{\mathcal{X}}\,(2_1)_{abcd}$, switch the type of the 4-th crossing, i. e., for example, replace $BBBB$ with $BBBA$, and vice versa. For $\widehat{\mathcal{X}}\,(0_1)_{abcd}$, switch the types of both the 3-rd and the 4-th crossing. Therefore, $\widehat{\mathcal{X}}\,(2_1)_{abcd} = a^{-8}x_{0,0,0,1}x_{2,1,0,-1} + a^{-4}x_{0,1,-2,0} + a^{-6}x_{2,1,0,0} + a^{-6}x_{2,1,0,-2}$, while $\widehat{\mathcal{X}}\,(2_1) = (a^{-4} + 2a^{-6})x + a^{-8}x^2$, and $\widehat{\mathcal{X}}\,(0_1)_{abcd} = x_{0,0,0,1}x_{2,1,0,-1} + x_{0,1,-2,0} + a^{-2}x_{2,1,0,0} + a^2x_{2,1,0,-2}$, while $\widehat{\mathcal{X}}\,(0_1) = (a^{-2} + 1 + a^2)x + x^2$.

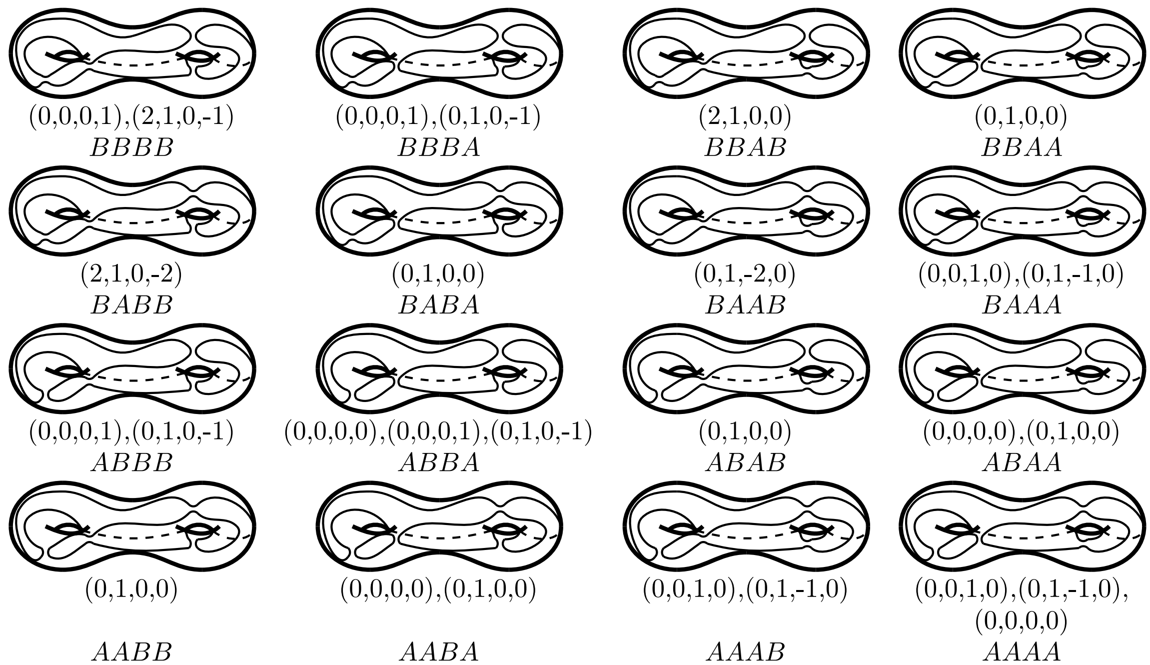

Fig. 6. States of the knot diagram 4_1 given in Fig. 5

Obviously, even $\widehat{\mathcal{X}}(\cdot)$ is enough to show that all three knots are not equivalent. Let us use $\widehat{\mathcal{X}}(\cdot)_{abcd}$ to determine if each of the knots 4_1, 2_1, and 0_1 can be realized in $T \times I$ or not. To this end, according to Lemma 3, we construct and solve a system of linear equations.

For the knot 4_1, we present only four equations to show that the unique solution is zero:

$$\begin{cases} c = 0 & \text{(given by (0,0,0,1))}, \\ -d = 0 & \text{(given by (0,0,1,0))}, \\ a - c = 0 & \text{(given by (0,1,0,-1))}, \\ a - 2 \cdot b = 0 & \text{(given by (2,1,0,0))}, \\ \dots \end{cases}$$

Therefore, according to Lemma 3, the knot 4_1 can not be realized in $T \times I$.

For the knots 2_1 and 0_1, such a system has the following form:

$$\begin{cases} c = 0 & \text{(given by (0,0,0,1))}, \\ a + 2 \cdot d = 0 & \text{(given by (0,1,-2,0))}, \\ a - 2 \cdot b = 0 & \text{(given by (2,1,0,0))}, \\ a - 2 \cdot b - c = 0 & \text{(given by (2,1,0,-1))}, \\ a - 2 \cdot b - 2 \cdot c = 0 & \text{(given by (2,1,0,-2))}. \end{cases}$$

The general solution is $(a, a/2, 0, -a/2)$. Therefore, according to Lemma 3, both knots 2_1 and 0_1 can be realized in $T \times I$, and we can use a particular solution $(2, 1, 0, -1)$ as the isotopy type of the destabilisation curve.

According to Case (i) of Lemma 4, we can reduce the isotopy type $(2, 1, 0, -1)$ of the destabilisation curve to $(0, 1, 0, -1)$. To this end, to take into account that all noncut curves are involved in the same transformations, we construct the matrix M and reduce the first row (multiply the 2-nd column by -2 and add to the 1-st column, i. e. double twist the left handle of the surface T_2 along its longitude in the opposite direction):

$$M = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}.$$

According to Lemma 5, the destabilisation performed along the noncut curve having the isotopy type $(0, 1, 0, -1)$ reduces the noncut curves involved in the terms of $\widehat{\mathcal{X}}\,(2_1)_{abcd}$ and $\widehat{\mathcal{X}}\,(0_1)_{abcd}$ as follows:

$$(0, 0, 0, 1) \to (0, 1),\ (-2, 1, -2, 0) \to (-2, 1),$$

$$(0, 1, 0, 0) \to (0, 1),\ (0, 1, 0, -2) \to (0, -1) \sim (0, 1).$$

In its turn, the destabilisation curve is reduced to a cut one in the torus T. Consequently, we replace $x_{0,1,0,-1}$ with $(-a^2 - a^{-2})$ in both $\widehat{\mathcal{X}}\,(2_1)_{abcd}$ and $\widehat{\mathcal{X}}\,(0_1)_{abcd}$. Therefore,

$$\widehat{\mathcal{X}}\,(2_1)_{abcd} \to \widetilde{\mathcal{X}}\,(2_1)_{pq} = a^{-8}x_{0,1} \cdot (-a^2 - a^{-2}) + a^{-4}x_{-2,1} + a^{-6}x_{0,1} + a^{-6}x_{0,1} =$$
$$= (a^{-6} - a^{-10})x_{0,1} + a^{-4}x_{-2,1}, \tag{4}$$

and $\widehat{\mathcal{X}}\,(0_1)_{abcd} \to \widetilde{\mathcal{X}}\,(0_1)_{pq} = x_{0,1} \cdot (-a^2 - a^{-2}) + x_{-2,1} + a^{-2}x_{0,1} + a^2x_{0,1} = x_{-2,1}$.

Since $\widetilde{\mathcal{X}}\,(2_1)_{pq}$ contains terms corresponding to the noncut curves having different isotopy types, the knot 2_1 can not be realised in the thickened annulus $\mathcal{A} \times I$. In its turn, the knot 0_1 can be realised in the thickened annulus $\mathcal{A} \times I$.

Note that the reduced $\widehat{\mathcal{X}}\,(2_1)_{abcd}$ (i. e., (4), where all terms of the form $x_{p,q}$ are considered to be equal to x) is equal to $\widetilde{\mathcal{X}}\,(2_1)$ given in [6], while $\widehat{\mathcal{X}}\,(2_1)$ has another value. Therefore, the direct comparison of $\widetilde{\mathcal{X}}\,(\cdot)$ and $\widehat{\mathcal{X}}\,(\cdot)$ does not allow to determine the possibility of destabilisation.

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ОБОБЩЕНИЯ ПОЛИНОМА КАУФМАНА ДЛЯ УЗЛОВ В УТОЛЩЁННОМ КРЕНДЕЛЕ РОДА 2

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Рассматриваются два инварианта, полученных в результате различных подходов к обобщению полинома Кауфмана на случай узлов в утолщённом кренделе рода 2. При построении первого обобщения все кривые разделяются только на два класса (разбивающие и неразбивающие). Полученный инвариант оказывается достаточно сильным, чтобы различить ряд узлов в утолщённом кренделе рода 2, хотя и не позволяет определить, может ли данный узел быть реализован в утолщённой поверхности меньшего рода. В свою очередь, второе обобщение дополнительно учитывает изотопические типы неразбивающих кривых, благодаря чему позволяет выполнить дестабилизацию, если только это возможно, и получить значение инварианта того же самого узла, реализованного в утолщённой поверхности меньшего рода. Вычислительные примеры показывают эффективность второго обобщения и неэффективность первого в смысле дестабилизации.

Ключевые слова: *полином Кауфмана, узел, утолщённый крендель рода 2, изотопические типы кривых.*

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