

# ENTIRE AND ANALYTICAL SOLUTIONS OF ABSTRACT DEGENERATE FRACTIONAL DIFFERENTIAL EQUATIONS

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We are primarily concerned with the study of entire and analytical solutions of abstract degenerate (multi-term) fractional differential equations with Caputo time-fractional derivatives. We also analyze systems of such equations and furnish several illustrative examples to demonstrate usage of obtained theoretical results.

**Ключевые слова:** *abstract degenerate differential equation, Volterra integro-differential equation, fractional differential equation, entire solution, analytical solution, well-posedness.*

## 1. Introduction and preliminaries

Fractional calculus has gained considerable popularity and importance during the past four decades, mainly due to its applications in diverse fields of science and engineering. Fairly complete information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting the references [1–8]. Various types of abstract degenerate Volterra integro-differential equations and abstract degenerate (multi-term) fractional differential equations have been recently considered in [9–15] (cf. also [16–24] for some other papers concerning the abstract degenerate differential equations).

It is well known that the study of entire solutions of abstract differential equations was initiated by R. deLaubenfels [25] in 1991. Concerning the theory of abstract differential equations with integer order derivatives, further contributions have been obtained by L. Autret [26], L. Autret and H.A. Emamirad [27], T.-J. Xiao and J. Liang [28], Y. Mishura and Y. Tomilov [29], and the author [7; 30]. The existence and uniqueness of entire and analytical solutions of the abstract non-degenerate time-fractional differential equations with Caputo derivatives have been investigated in [31; 32]. In a joint research paper with V. Fedorov [33], the author has recently considered a class of abstract degenerate multi-term fractional differential equations in locally convex spaces, pointing out that the methods proposed in [28] (cf. also [34, Section 4.4, pp. 167–175]), [31] and [33, Remark 2.2 (x), (xi)] can serve one to prove some results on the existence and uniqueness of entire solutions of degenerate multi-term differential equations with integer order derivatives (cf. [21, Chapter 4] for some basic results on the entire groups of solving operators for abstract degenerate differential equations of first order). Motivated primarily by this fact, in the second section of paper we consider the existence and uniqueness of entire and analytical solutions to (systems) of degenerate multi-term fractional differential equations with Caputo derivatives. It should also be noticed that in Subsection 2.1 we initiate the analysis of existence

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and uniqueness of entire and analytical solutions of some very important degenerate equations of mathematical physics in  $L^p$  type spaces.

We use the standard notation throughout the paper. Unless specified otherwise, we assume that  $X$  is a Hausdorff sequentially complete locally convex space over the field of complex numbers. We use the shorthand SCLCS to denote such a space. By  $L(X)$  we denote the space consisting of all continuous linear mappings from  $X$  into  $X$ . By  $\otimes$  we denote the fundamental systems of seminorms which defines the topology of  $X$ . The Hausdorff locally convex topology on  $L(X)$  is defined in the usual way (see [8, Section 1.1]). Let us recall that the space  $L(X)$  is sequentially complete provided that  $X$  is barreled [35]. If  $A$  is a linear operator acting on  $X$ , then the domain, kernel space and range of  $A$  will be denoted by  $D(A)$ ,  $N(A)$  and  $R(A)$ , respectively. Since no confusion seems likely, we will identify  $A$  with its graph. The symbol  $I$  stands for the identity operator on  $X$ . If  $C \in L(X)$  and  $A$  is a closed linear operator acting on  $X$ , then we define the  $C$ -resolvent set of  $A$ ,  $\rho_C(A)$  for short, by  $\rho_C(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X)\}$ ;  $\rho(A) \equiv \rho_I(A)$ . If  $V$  is a general topological vector space, then a function  $f : \Omega \rightarrow V$ , where  $\Omega$  is an open non-empty subset of  $\mathbb{C}$ , is said to be analytic iff it is locally expressible in a neighborhood of any point  $z \in \Omega$  by a uniformly convergent power series with coefficients in  $V$ . We refer the reader to [36] and [8, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our approach the space  $X$  is sequentially complete, so that the analyticity of a mapping  $f : \Omega \rightarrow X$  is equivalent with its weak analyticity.

By  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  we denote the Fourier transform on  $\mathbb{R}^n$  and its inverse transform, respectively. Given  $\theta \in (0, \pi]$  in advance, define  $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$ . Further on,  $[\beta] := \sup\{k \in \mathbb{Z} : k \leq \beta\}$ ,  $[\beta] := \inf\{k \in \mathbb{Z} : \beta \leq k\}$  ( $\beta \in \mathbb{R}$ ),  $\mathbb{N}_n := \{1, \dots, n\}$  and  $\mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\}$  ( $n \in \mathbb{N}$ ). The Gamma function is denoted by  $\Gamma(\cdot)$  and the principal branch is always used to take the powers; the convolution like mapping  $*$  is given by  $f * g(t) := \int_0^t f(t-s)g(s) ds$ . Set  $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ,  $0^\zeta := 0$  ( $\zeta > 0, t > 0$ ), and  $g_0(t) :=$  the Dirac  $\delta$ -distribution. For a number  $\zeta > 0$  given in advance, the Caputo fractional derivative  $\mathbf{D}_t^\zeta u$  [4; 8] is defined for those functions  $u \in C^{[\zeta]-1}([0, \infty) : X)$  for which  $g_{[\zeta]-\zeta} * (u - \sum_{j=0}^{[\zeta]-1} u^{(j)}(0)g_{j+1}) \in C^{[\zeta]}([0, \infty) : X)$ , by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^{[\zeta]}}{dt^{[\zeta]}} \left[ g_{[\zeta]-\zeta} * \left( u - \sum_{j=0}^{[\zeta]-1} u^{(j)}(0)g_{j+1} \right) \right].$$

The Mittag-Leffler function  $E_{\beta,\gamma}(z)$  ( $\beta > 0, \gamma \in \mathbb{R}$ ) is defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

In this place, we assume that  $1/\Gamma(\beta k + \gamma) = 0$  if  $\beta k + \gamma \in -\mathbb{N}_0$ . Set, for short,  $E_\beta(z) := E_{\beta,1}(z)$ ,  $z \in \mathbb{C}$ . The asymptotic behaviour of the entire function  $E_{\beta,\gamma}(z)$  is given in the following auxiliary lemma (see e.g. [8, Section 1.3]):

**Lemma 1.** *Let  $0 < \sigma < \pi/2$ . Then, for every  $z \in \mathbb{C} \setminus \{0\}$  and  $l \in \mathbb{N} \setminus \{1\}$ ,*

$$E_{\beta,\gamma}(z) = \frac{1}{\beta} \sum_s Z_s^{1-\gamma} e^{Z_s} - \sum_{j=1}^{l-1} \frac{z^{-j}}{\Gamma(\gamma - \beta j)} + O(|z|^{-l}), \quad |z| \rightarrow \infty,$$

where  $Z_s$  is defined by  $Z_s := z^{1/\beta} e^{2\pi i s/\beta}$  and the first summation is taken over all those integers  $s$  satisfying  $|\arg(z) + 2\pi s| < \beta(\sigma + \pi/2)$ .

For further information about the Mittag-Leffler functions and the abstract Volterra integro-differential equations in Banach and locally convex spaces, the reader may consult [1; 4; 8] and references cited there.

Assume that  $n \in \mathbb{N}$  and  $iA_j$ ,  $1 \leq j \leq n$ , are commuting generators of bounded  $C_0$ -groups on a Banach space  $X$ . Set  $A := (A_1, \dots, A_n)$  and  $A^\eta := A_1^{\eta_1} \cdots A_n^{\eta_n}$  for any  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{N}_0^n$ ; denote by  $\mathcal{D}(\mathbb{R}^n)$  and  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space of  $C^\infty(\mathbb{R}^n)$ -functions with compact support and the Schwartz space of rapidly decreasing functions on  $\mathbb{R}^n$ , respectively (in the sequel, the meaning of symbol  $A$  will be clear from the context). Let  $k = 1 + \lfloor n/2 \rfloor$ . For every  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  and  $u \in \mathcal{FL}^1(\mathbb{R}^n) = \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$ , we set  $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$ ,  $(\xi, A) := \sum_{j=1}^n \xi_j A_j$  and  $u(A)x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi) e^{-i(\xi, A)} x d\xi$ ,  $x \in X$ . Then  $u(A) \in L(X)$ ,  $u \in \mathcal{FL}^1(\mathbb{R}^n)$  and there exists a finite constant  $M \geq 1$  such that  $\|u(A)\| \leq M \|\mathcal{F}^{-1}u\|_{L^1(\mathbb{R}^n)}$ ,  $u \in \mathcal{FL}^1(\mathbb{R}^n)$ . Let  $N \in \mathbb{N}$ , and let  $p(x) = \sum_{|\eta| \leq N} a_\eta x^\eta$ ,  $x \in \mathbb{R}^n$  be a complex polynomial. Then we define  $p(A) := \sum_{|\eta| \leq N} a_\eta A^\eta$  and  $X_0 := \{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n), x \in X\}$ . We know that the operator  $p(A)$  is closable and that the following holds:

$$(\triangleright) \overline{X_0} = X, X_0 \subseteq \bigcap_{\eta \in \mathbb{N}_0^n} D(A^\eta), \overline{p(A)|_{X_0}} = \overline{p(A)} \text{ and } \phi(A)p(A) \subseteq p(A)\phi(A) = (\phi p)(A), \phi \in \mathcal{S}(\mathbb{R}^n).$$

Denote by  $\mathbb{C}^{m,m}$  the ring of  $m \times m$  matrices over  $\mathbb{C}$ ;  $I_m$  stands for the identity matrix of format  $m \times m$  ( $m \in \mathbb{N}$ ). If  $P(x) = [p_{ij}(x)]$  is an  $m \times m$  matrix of polynomials of  $x \in \mathbb{R}^n$ , then there exist  $d \in \mathbb{N}$  and matrices  $P_\eta \in \mathbb{C}^{m,m}$  such that  $P(x) = \sum_{|\eta| \leq d} P_\eta x^\eta$ ,  $x \in \mathbb{R}^n$ . Then the operator  $P(A) := \sum_{|\eta| \leq d} P_\eta A^\eta$  is closable on  $X^m$ . For further information concerning the functional calculus for commuting generators of bounded  $C_0$ -groups, see [8; 37–39].

The proof of following auxiliary lemma, which is probably known in the existing literature, is included for the sake of completeness.

**Lemma 2.** *Suppose that  $1 \leq p < \infty$ ,  $n \in \mathbb{N}$  and  $X := L^p(\mathbb{R}^n)$ . Denote by  $\Delta_{p,n}$  the operator  $\Delta$  acting on  $X$  with its maximal distributional domain. Then  $\Delta_{p,n}$  is injective.*

*Proof.* If  $1 < p < \infty$ , then the statement immediately follows from the fact that the operator  $-\Delta_{p,n}$  is non-negative, with dense domain and range (cf. [40, pp. 256, 266]). Suppose now that  $p = 1$  and  $\Delta_{p,n}f = 0$  for some  $f \in X$ . Then [40, Lemma 3.2] implies that, for every  $\varphi \in \mathcal{D}(\mathbb{R}^n)$  and for every multi-index  $\eta \in \mathbb{N}_0^n$ , the function  $\varphi * f$  belongs to the space  $\mathcal{T}$  consisting of those  $C^\infty(\mathbb{R}^n)$ -functions whose any partial derivative belongs to  $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Since  $\Delta_{p,n}(\varphi * f) = \varphi * \Delta_{p,n}f = 0$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$  and the operator  $\Delta_{\mathcal{T}}$  is injective by [40, Remark 3.3], we have that  $\varphi * f = 0$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ . Hence,  $f = 0$ .  $\square$

## 2. Formulation and proof of main results.

### Examples and applications

Before stating our first main result, we need to repeat some notations and preliminaries from [33]. Suppose that  $n \in \mathbb{N}$ ,  $0 < \zeta \leq 2$ ,  $q_0, q_1, \dots, q_n$  are given non-negative integers satisfying  $q_0 = 0$  and  $0 < q_1 \leq q_2 \leq \dots \leq q_n$ . Let  $A_0, A_1, \dots, A_{n-1}, A_n$  be closed linear operators acting on  $X$ . Set  $A_n := B$ ,  $T_i u(t) := A_i (\mathbf{D}_t^\zeta)^{q_i} u(t)$ ,  $t \geq 0$ ,  $i \in \mathbb{N}_n^0$  and

$$P_\lambda := \lambda^{q_n \zeta} B + \sum_{i=0}^{n-1} \lambda^{q_i \zeta} A_i, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Of concern is the following abstract degenerate multi-term Cauchy problem:

$$\sum_{i=0}^n T_i u(t) = 0, \quad t \geq 0, \quad (1)$$

accompanied with the following initial conditions:

$$\begin{aligned} \left( (\mathbf{D}_t^\zeta)^j u(t) \right)_{t=0} &= u_j, \quad j \in \mathbb{N}_{q_n-1}^0, \quad \text{if } \zeta \in (0, 1], \quad \text{resp.}, \\ \left( (\mathbf{D}_t^\zeta)^j u(t) \right)_{t=0} &= u_j, \quad j \in \mathbb{N}_{q_n-1}^0; \quad \left( \frac{d}{dt} (\mathbf{D}_t^\zeta)^j u(t) \right)_{t=0} = v_j, \quad j \in \mathbb{N}_{q_n-1}^0, \quad \text{if } \zeta \in (1, 2]. \end{aligned} \quad (2)$$

In [33], we have considered the abstract Cauchy problem (1), (2) with  $0 < \zeta \leq 1$ . The notion of a strong solution of problem (1), (2), introduced in the first part of following definition, coincides with the corresponding notion introduced in [33, Definition 1] in the case that  $0 < \zeta \leq 1$ .

**Definition 1.** (i) A function  $u \in C([0, \infty) : X)$  is said to be a *strong solution of problem (1), (2)* iff the term  $T_i u(t)$  is well defined and continuous for any  $t \geq 0$ ,  $i \in \mathbb{N}_n^0$ , and (1), (2) holds identically on  $[0, \infty)$ .

(ii) A function  $u \in C([0, \infty) : X)$  is said to be an *entire solution of problem (1), (2)* iff  $u(\cdot)$  is a strong solution of (1), (2) and it can be analytically extended to the whole complex plane, as well as any of the terms  $A_i u^{(p)}(\cdot)$  ( $0 \leq i \leq n$ ,  $p \in \mathbb{N}_0$ ) can be analytically extended to the whole complex plane.

(iii) A function  $u \in C([0, \infty) : X)$  is said to be an *analytical solution of problem (1), (2)* on the region  $\mathbb{C} \setminus (-\infty, 0]$  iff  $u(\cdot)$  is a strong solution of (1), (2) and it can be extended to the whole complex plane, analytically on the region  $\mathbb{C} \setminus (-\infty, 0]$  and continuously on the region  $\mathbb{C} \setminus (-\infty, 0)$ , as well as any of the terms  $A_i (\mathbf{D}_t^\zeta)^p u(t)$  ( $0 \leq i \leq n$ ,  $p \in \mathbb{N}_0$ ,  $t \geq 0$ ) is well defined and can be extended to the whole complex plane, analytically on the region  $\mathbb{C} \setminus (-\infty, 0]$  and continuously on the region  $\mathbb{C} \setminus (-\infty, 0)$ .

Set  $S_\omega := \{j \in \mathbb{N}_n^0 : q_j - 1 \geq \omega\}$  ( $\omega \in \mathbb{N}_{q_n-1}^0$ ). We need to introduce the following condition:

$$-\infty < \nu' < \min_{\omega \in \mathbb{N}_{q_n-1}^0} \left[ - \left( q_n - 1 - \omega + \max \{ q_j : j \in \mathbb{N}_n^0 \setminus S_\omega \} \right) \right]. \quad (3)$$

Then  $n \in S_\omega$  for all  $\omega \in \mathbb{N}_{q_n-1}^0$ , and (3) holds provided that  $-\infty < \nu' < 1 - q_n$ .

Now we are ready to formulate the following theorem.

**Theorem 2.** Suppose that the operator  $C \in L(X)$  is injective,  $CA_i \subseteq A_i C$ ,  $i \in \mathbb{N}_n^0$ ,  $0 < \zeta \leq 2$ ,  $\phi \in (-\pi, \pi]$ ,  $\theta \in (\pi - \pi\zeta, \pi - (\pi\zeta)/2)$ ,  $a > r > 0$  and  $\nu'$  satisfies (3). Assume, further, that the following holds:

(i) the operator family  $\{(1 + |\lambda|)^{-\nu'} P_{\lambda^{1/\zeta}}^{-1} C : \lambda \in e^{i\phi} \Sigma_{(\zeta\pi/2)+\theta}, |\lambda| \geq r\} \subseteq L(X)$  is both equicontinuous and strongly continuous;

(ii) for every  $x \in X$  and  $i \in \mathbb{N}_{n-1}^0$ , the mapping  $\lambda \mapsto A_i P_{\lambda^{1/\zeta}}^{-1} Cx$ ,  $\lambda \in e^{i\phi} \Sigma_{(\zeta\pi/2)+\theta}$ ,  $|\lambda| \geq r$  is continuous and there exists  $v_i \in \mathbb{N}$  such that the operator family  $\{(1 + |\lambda|)^{-v_i} A_i P_{\lambda^{1/\zeta}}^{-1} C : \lambda \in e^{i\phi} \Sigma_{(\zeta\pi/2)+\theta}, |\lambda| \geq r\} \subseteq L(X)$  is equicontinuous.

Denote by  $\mathfrak{W}$  ( $\mathfrak{W}_e$ ) the subspace of  $X^{q_n}$ , resp.  $X^{2q_n}$ , consisting of all initial values  $(u_0, \dots, u_{q_n-1}) \in X^{q_n}$ , resp.  $(u_0, \dots, u_{q_n-1}; v_0, \dots, v_{q_n-1}) \in X^{2q_n}$ , subjected to some analytical solution  $u(\cdot)$  of problem (1), (2) on the region  $\mathbb{C} \setminus (-\infty, 0]$  (entire solution  $u(\cdot)$  of problem (1), (2)). Then  $\mathfrak{W}$  is dense in  $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$  for the topology of  $X^{q_n}$ , provided that  $0 < \zeta < 1$ , resp.  $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$  for the topology of  $X^{2q_n}$ , provided

that  $1 < \zeta < 2$ ; furthermore,  $\mathfrak{W}_e$  is dense in  $(C(\bigcap_{j=0}^n D(A_j)))^{q_n}$  for the topology of  $X^{q_n}$ , provided that  $\zeta = 1$ , resp.  $(C(\bigcap_{j=0}^n D(A_j)))^{2q_n}$  for the topology of  $X^{2q_n}$ , provided that  $\zeta = 2$ .

To prove Theorem 2, we need the following lemma (cf. also [23, Lemma 1.1, Theorem 1.1]).

**Lemma 3.** *Let  $x \in X$ . Then the mapping  $\lambda \mapsto P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} Cx$ ,  $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$ ,  $|\lambda| > r$ , is analytic.*

*Proof.* Without loss of generality, we may assume that  $q_i = i$  ( $i \in \mathbb{N}_n^0$ ),  $\zeta = 1$  and  $\phi = 0$ . Clearly, (ii) holds for every  $x \in X$  and  $i \in \mathbb{N}_n^0$ . Furthermore, the following analogon of the Hilbert resolvent equation holds:

$$\begin{aligned} P_\lambda^{-1} C^2 x - P_z^{-1} C^2 x &= (z - \lambda) P_\lambda^{-1} C \times \\ &\times \left[ \sum_{k=1}^{n-1} \binom{n}{k} (z - \lambda)^{k-1} \lambda^{n-k} B + \sum_{k=1}^{n-2} \binom{n-1}{k} (z - \lambda)^{k-1} \lambda^{n-1-k} A_{n-1} + \dots + A_1 \right] \times \\ &\times P_z^{-1} Cx, \text{ provided } \lambda, z \in \Sigma_{(\zeta\pi/2)+\theta} \text{ and } |\lambda|, |z| > r. \end{aligned}$$

This implies that the mapping  $\lambda \mapsto P_\lambda^{-1} C^2 x$ ,  $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$ ,  $|\lambda| > r$ , is weakly analytic and therefore analytic, as well as that

$$\frac{d}{d\lambda} \langle x^*, P_\lambda^{-1} C^2 x \rangle = - \langle x^*, P_\lambda^{-1} [n\lambda^{n-1} B + (n-1)\lambda^{n-2} A_{n-1} + \dots + A_1] P_\lambda^{-1} Cx \rangle,$$

provided  $x^* \in X^*$ ,  $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$  and  $|\lambda| > r$ . Using the Morera theorem and the observation from [41, Remark 2.7], the above implies that the mapping  $\lambda \mapsto P_\lambda^{-1} Cx$ ,  $\lambda \in \Sigma_{(\zeta\pi/2)+\theta}$ ,  $|\lambda| > r$ , is analytic, as claimed.  $\square$

Now we can proceed to the proof of Theorem 2.

*Proof.* Suppose first  $0 < \zeta \leq 1$ . Clearly,  $(\zeta\pi/2) + \theta < \pi$ ,  $\pi\zeta/2 > \pi - (\pi\zeta/2) - \theta$  and we can find a number  $b \in \mathbb{R}$  satisfying

$$1 < b < \frac{\pi\zeta/2}{\pi - (\pi\zeta/2) - \theta}.$$

Denote by  $\Gamma$  the upwards oriented boundary of the region  $\{\lambda \in \Sigma_{(\zeta\pi/2)+\theta} : |\lambda| \geq r\}$ . Let  $\Omega$  be the open region on the left of  $\Gamma$ . Then there exists a sufficiently large number  $R > 0$  such that  $a - \lambda \in \Sigma_{\pi - (\pi\zeta/2) - \theta}$  for all  $\lambda \in \Omega \cup \Gamma$  with  $|\lambda| \geq R$ . This implies  $|e^{-\epsilon(a-\lambda)^{b/\zeta}}| = e^{-\epsilon \operatorname{Re}((a-\lambda)^{b/\zeta})} \leq e^{-\epsilon|a-\lambda|^{b/\zeta} \cos(b\zeta^{-1}(\pi - (\pi\zeta/2) - \theta))}$ , provided  $\epsilon > 0$ ,  $\lambda \in \Omega \cup \Gamma$  and  $|\lambda| \geq R$ . Keeping in mind Lemma 1, we obtain the existence of a constant  $c'_\zeta > 0$  such that  $|E_\zeta(z^\zeta \lambda e^{i\phi})| \leq E_\zeta(|z|^\zeta |\lambda|) \leq c'_\zeta e^{|z||\lambda|^{1/\zeta}}$  for all  $z \in \mathbb{C}$  and  $\lambda \in \mathbb{C}$ . Hence, there exists a constant  $c_\zeta > 0$  such that

$$\left| e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(z^\zeta \lambda e^{i\phi}) \right| \leq c_\zeta e^{-\epsilon|a-\lambda|^{b/\zeta} \cos(b\zeta^{-1}(\pi - (\pi\zeta/2) - \theta)) + |z||\lambda|^{1/\zeta}}, \tag{4}$$

for any  $z \in \mathbb{C}$ ,  $\epsilon > 0$  and  $\lambda \in \Omega$ . Suppose now that  $x_w \in \bigcap_{j=0}^n D(A_j)$  for all  $w \in \mathbb{N}_{q_n-1}^0$ . Then (i) and the estimate (4) enable us to define the function  $z \mapsto u_\epsilon(z)$ ,  $z \in \mathbb{C}$ , for any  $\epsilon > 0$ , by

$$u_\epsilon(z) := \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \frac{1}{2\pi i} \int_\Gamma e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(z^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda.$$

It can be simply verified that the mapping  $z \mapsto u_\epsilon(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0)$  is continuous ( $\epsilon > 0$ ). Using Lemma 1 and the proof of [33, Theorem 2], it readily follows that the mapping  $z \mapsto u_\epsilon(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$ , is analytic ( $\epsilon > 0$ ), with

$$u'_\epsilon(z) = \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} z^{\zeta-1} E_{\zeta, \zeta}(z^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{q_j-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda, \quad (5)$$

for any  $\epsilon > 0$  and  $z \in \mathbb{C} \setminus (-\infty, 0]$ ; furthermore, the mapping  $z \mapsto u_\epsilon(z)$ ,  $z \in \mathbb{C}$  is entire provided  $\epsilon > 0$ ,  $\zeta = 1$  and, in this case, (5) holds for any  $\epsilon > 0$  and  $z \in \mathbb{C}$ . The proof of [33, Theorem 2] also shows that the term  $(\mathbf{D}_t^\zeta)^p u_\epsilon(t)$ ,  $t \geq 0$ , is well defined for any  $p \in \mathbb{N}_0$  and  $\epsilon > 0$ , with

$$(\mathbf{D}_t^\zeta)^p u_\epsilon(t) = \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{p+q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda; \quad (6)$$

cf. also the formula [4, (1.25)]. Combined with the Cauchy theorem, (ii) and Lemma 1, the above implies that the term  $A_i (\mathbf{D}_t^\zeta)^p u_\epsilon(t)$  is well defined for  $t \geq 0$ ,  $i \in \mathbb{N}_n^0$ ,  $p \in \mathbb{N}_0$  and  $\epsilon > 0$ , with

$$A_i (\mathbf{D}_t^\zeta)^p u_\epsilon(t) = \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{p+q_j-1-w} A_i P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda.$$

This implies that, for every  $\epsilon > 0$ , any of the terms  $A_i (\mathbf{D}_t^\zeta)^p u_\epsilon(\cdot)$  ( $0 \leq i \leq n$ ,  $p \in \mathbb{N}_0$ ) can be extended to the whole complex plane, analytically on the region  $\mathbb{C} \setminus (-\infty, 0]$  and continuously on the region  $\mathbb{C} \setminus (-\infty, 0)$ ; we only need to replace the variable  $t \geq 0$ , appearing in the above formula, with the variable  $z \in \mathbb{C}$ . Furthermore,

$$\begin{aligned} \sum_{i=0}^n A_i (\mathbf{D}_t^\zeta)^{q_i} u_\epsilon(t) &= \\ &= \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \sum_{i=0}^n \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{q_i+q_j-1-w} A_i P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda = \\ &= \sum_{w=0}^{q_n-1} \sum_{j \in S_w} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{q_j-1-w} C A_j x_w d\lambda = 0, \quad t \geq 0, \quad \epsilon > 0, \end{aligned}$$

so that for each  $\epsilon > 0$  the mapping  $t \mapsto u_\epsilon(t)$ ,  $t \geq 0$ , is an analytical solution of problem (1), (2) on the region  $\mathbb{C} \setminus (-\infty, 0]$  (entire solution of problem (1), (2), provided that  $\zeta = 1$ ). Let  $u_l^\epsilon = ((\mathbf{D}_t^\zeta)^l u_\epsilon(t))_{t=0}$ ,  $l \in \mathbb{N}_{q_n-1}^0$  ( $\epsilon > 0$ ). Now we will prove that  $(u_0^\epsilon, \dots, u_{q_n-1}^\epsilon)$  converges to  $e^{-i\phi}(C x_0, \dots, C x_{q_n-1})$  as  $\epsilon \rightarrow 0+$ , for the topology of  $X^{q_n}$  (cf. also [33, Remark 1 (x)]). Let  $\omega \in \mathbb{N}_{q_n-1}^0$  and  $l \in \mathbb{N}_{q_n-1}^0$  be fixed. Keeping in mind (6), it suffices to prove that the following holds:

$$\lim_{\epsilon \rightarrow 0+} \sum_{j \in S_w} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} (\lambda e^{i\phi})^{l+q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda = e^{-i\phi} \delta_{\omega l} C x_w,$$

i.e., that

$$\lim_{\epsilon \rightarrow 0+} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} (\lambda e^{i\phi})^{l-1-w} \left[ C x_w - \sum_{j \in \mathbb{N}_n^0 \setminus S_w} (\lambda e^{i\phi})^{q_j} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w \right] d\lambda = e^{-i\phi} \delta_{\omega l} C x_w, \quad (7)$$

where  $\delta_{\omega l}$  denotes the Kronecker delta. Since  $|e^{-\epsilon(a-\lambda)^{b/\zeta}}| \leq 1$ ,  $\lambda \in \Gamma$ ,  $\epsilon > 0$ , (3) and (i) holds, we have that there exists  $\sigma > 0$  such that

$$\left| e^{-\epsilon(a-\lambda)^{b/\zeta}} (\lambda e^{i\phi})^{l-1-w} (\lambda e^{i\phi})^{q_j} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w \right| \leq \text{Const. } |\lambda|^{-1-\sigma},$$

for any  $\lambda \in \Gamma$ ,  $\epsilon > 0$  and  $j \in \mathbb{N}_n^0 \setminus S_\omega$ . Applying the dominated convergence theorem, Lemma 3 and the Cauchy theorem, we get that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} (\lambda e^{i\phi})^{l-1-w} \sum_{j \in \mathbb{N}_n^0 \setminus S_\omega} (\lambda e^{i\phi})^{q_j} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda = \\ = \frac{1}{2\pi i} \int_{\Gamma} (\lambda e^{i\phi})^{l-1-w} \sum_{j \in \mathbb{N}_n^0 \setminus S_\omega} (\lambda e^{i\phi})^{q_j} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda = 0. \end{aligned}$$

Taking into account the last formula on p. 170 of [34], it readily follows that (7) holds good. The proof of theorem is thereby complete in the case that  $0 < \zeta \leq 1$ . Suppose now  $1 < \zeta \leq 2$ . Then it is not difficult to show that there exists a finite constant  $d_\zeta > 0$  such that the function  $F_\lambda(z) := z E_{\zeta,2}(z^\zeta \lambda e^{i\phi})$ ,  $z \in \mathbb{C}$  ( $\lambda \in \mathbb{C}$ ) satisfies  $F'_\lambda(z) = E_\zeta(z^\zeta \lambda e^{i\phi})$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$  and  $|F_\lambda(z)| \leq d_\zeta(1 + |z|)e^{|z||\lambda|^{1/\zeta}}$ ,  $z \in \mathbb{C}$  ( $\lambda \in \mathbb{C}$ ). Since for any function  $u \in C^1([0, \infty) : X)$  with  $u'(0) = 0$  we have  $\mathbf{D}_t^\zeta(g_1 * u)(t) = (g_1 * \mathbf{D}_t^\zeta u)(t)$ ,  $t \geq 0$ , provided in addition that the term  $\mathbf{D}_t^\zeta u(t)$  is defined for  $t \geq 0$ , it readily follows that  $\mathbf{D}_t^\zeta F_\lambda(t) = (g_1 * \mathbf{D}_t^\zeta E_\zeta(\cdot^\zeta \lambda e^{i\phi}))(t) = \lambda e^{i\phi}(g_1 * E_\zeta(\cdot^\zeta \lambda e^{i\phi}))(t) = \lambda e^{i\phi} F_\lambda(t)$ ,  $t \geq 0$  ( $\lambda \in \mathbb{C}$ ). Let  $x_w, y_w \in \bigcap_{j=0}^n D(A_j)$  for all  $w \in \mathbb{N}_{q_n-1}^0$ . Define now the solution  $u_\epsilon(\cdot)$  by

$$\begin{aligned} u_\epsilon(z) := \sum_{w=0}^{q_n-1} \sum_{j \in S_\omega} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(z^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda + \\ + \sum_{w=0}^{q_n-1} \sum_{j \in S_\omega} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} F_\lambda(z) (\lambda e^{i\phi})^{q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j y_w d\lambda, \end{aligned}$$

for any  $z \in \mathbb{C}$  and  $\epsilon > 0$ . It is not difficult to prove that, for every  $p \in \mathbb{N}_0$ ,  $t \geq 0$  and  $\epsilon > 0$ , the following holds:

$$\begin{aligned} (\mathbf{D}_t^\zeta)^p u_\epsilon(t) = \sum_{w=0}^{q_n-1} \sum_{j \in S_\omega} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{p+q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda + \\ + \sum_{w=0}^{q_n-1} \sum_{j \in S_\omega} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} F_\lambda(t) (\lambda e^{i\phi})^{p+q_j-1-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j y_w d\lambda \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} (\mathbf{D}_t^\zeta)^p u_\epsilon(t) = \sum_{w=0}^{q_n-1} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} t^{\zeta-1} E_{\zeta,\zeta}(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{p+q_j-w} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j x_w d\lambda + \\ + \sum_{w=0}^{q_n-1} \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon(a-\lambda)^{b/\zeta}} E_\zeta(t^\zeta \lambda e^{i\phi}) (\lambda e^{i\phi})^{p+q_j-w-1} P_{(\lambda e^{i\phi})^{1/\zeta}}^{-1} C A_j y_w d\lambda. \end{aligned}$$

The remaining part of proof of theorem can be deduced by repeating almost literally the arguments that we have already used in the case that  $0 < \zeta \leq 1$ .  $\square$

**Remark 1.** (i) Theorem 2 seems to be new and not considered elsewhere provided that  $B \neq I$  or  $\zeta \neq 1$ .

(ii) As explained in [33, Remark 1 (iv)], the operator  $(\mathbf{D}_s^\zeta)^p$  and the operator  $\mathbf{D}_s^{\zeta p}$  can be completely different provided that  $\zeta \in (0, 2) \setminus \{1\}$  and  $p \in \mathbb{N} \setminus \{1\}$ , which clearly implies that we have to make a strict distinction between them. Observe also that Theorem 2 can be reformulated in the case that  $\zeta > 2$  (cf. also [31, Theorem 2.1, Theorem 2.2]) and that we can prove a similar result on the existence and uniqueness of entire and analytical solutions of problem  $(DFP)_R$  considered in [33]; we leave the reader to make this precise.

(iii) The notion of an entire solution of the abstract Cauchy problem  $(ACP_n)$ , introduced in [28, Definition 1.1], is slightly different from the corresponding notion introduced in Definition 1 (ii). Strictly speaking, if  $u(\cdot)$  is an entire solution of the abstract Cauchy problem  $(ACP_n)$  in the sense of Definition 1 (ii), then  $u(\cdot)$  is an entire solution of problem  $(ACP_n)$  in the sense of [28, Definition 1.1]. The converse statement holds provided that for each index  $i \in \mathbb{N}_{n-1}$  the initial values  $u_0, \dots, u_{i-1}$  belong to  $D(A_i)$ .

(iv) The uniqueness of analytical solutions of problem (1), (2) on the region  $\mathbb{C} \setminus (-\infty, 0]$  can be proved as follows. Let  $u(\cdot)$  be an analytical solution of problem (1), (2) on the region  $\mathbb{C} \setminus (-\infty, 0]$ , with the initial values  $u_j$ , resp.  $v_j$ , being zeroes ( $0 \leq j \leq q_n - 1$ ). Then the choice of initial values in (2) enables us to integrate the equation (1)  $(q_n \zeta)$ -times by using the formula [4, (1.21)]. Keeping in mind the analyticity of  $u(\cdot)$ , we easily infer that for each  $i \in \mathbb{N}_{n-1}^0$  the mappings  $z \mapsto Bu(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0)$  ( $z \mapsto Bu(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$ ), and  $z \mapsto (g_{(q_n - q_i)\zeta} * A_i u)(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0)$  ( $z \mapsto (g_{(q_n - q_i)\zeta} * A_i u)(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$ ), are well defined and continuous (analytical), as well as that

$$Bu(te^{i\gamma}) + \sum_{i=0}^{n-1} \int_0^{te^{i\gamma}} g_{(q_n - q_i)\zeta}(s) A_i u(te^{i\gamma} - s) ds = 0, \quad t \geq 0, \gamma \in (-\pi, \pi),$$

i.e., that

$$Bu(te^{i\gamma}) + \sum_{i=0}^{n-1} (e^{i\gamma})^{(q_n - q_i)\zeta} \int_0^t g_{(q_n - q_i)\zeta}(t-s) A_i u(se^{i\gamma}) ds = 0, \quad t \geq 0, \gamma \in (-\pi, \pi). \quad (8)$$

It is clear that there exists  $\gamma \in (-\pi, \pi)$  such that  $2re^{-i\gamma\zeta} \in e^{i\phi\zeta} \Sigma_{(\zeta\pi/2)+\theta}$ . Setting  $\phi' := \gamma\zeta$ ,  $u_\gamma(t) := u(te^{i\gamma})$ ,  $t \geq 0$ , we obtain that  $u_\gamma \in C([0, \infty) : X)$  and

$$e^{-iq_n\phi'} Bu_\gamma(t) + \sum_{i=0}^{n-1} e^{-iq_i\phi'} A_i (g_{(q_n - q_i)\zeta} * u_\gamma)(t) = 0, \quad t \geq 0.$$

On the other hand,

$$P_{(\lambda e^{-i\phi'})^{1/\zeta}} = \lambda^{q_n} e^{-iq_n\phi'} B + \sum_{i=0}^{n-1} \lambda^{q_i} e^{-iq_i\phi'} A_i, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Using the previous two equalities and [14, Theorem 2.2] (applied to the operators  $e^{-iq_i\phi'} A_i$  in place of the operators  $A_i$  appearing in the formulation of this theorem), we get that  $u_\gamma(t) = 0$ ,  $t \geq 0$ , which clearly implies that  $u(z) = 0$ ,  $z \in \mathbb{C} \setminus (-\infty, 0)$ .

It is also worth noting that Theorem 2 is an extension of [28, Theorem 2.1] (cf. also [34, Theorem 4.2, p. 168]), where it has been assumed that  $B = C = I$ ,  $\zeta = 1$ ,  $X$  is a



Banach space and  $\bigcap_{j=0}^n D(A_j)$  is dense in  $X$  (in our opinion, the strong continuity in (ii) is very important for the validity of Theorem 2 and cannot be so simply neglected here (cf. [34, (4.8), p. 169]); also, it ought to be observed that Lemma 3 is very important for filling some absences in the proof of [34, Theorem 4.2], appearing on the lines 1–6, p. 171 in [34], where the Cauchy formula has been used by assuming the analyticity of mapping  $\lambda \mapsto R_{e^{i\phi}\lambda}$ ,  $\lambda \in \Sigma_{(\pi/2)+\theta}$ ,  $|\lambda| > r$ , a priori); observe also that, in the concrete situation of abstract Cauchy problem  $(ACP_n)$ , our estimate on the growth rate of  $P_{e^{i\phi}}^{-1}$ . (cf. the equation (3) with  $\nu' < -(n - 1)$ ) is slightly better than the corresponding estimate [34, (4.2)], where it has been required that  $\nu' \leq -n$ . If  $B = C = I$  and  $\zeta = 2$ , then Theorem 2 strengthens [28, Theorem 2.1] in a drastic manner. Speaking-matter-of-factly, our basic requirement in (i) is that the operator  $P_\lambda^{-1} = (\lambda^{2n} + \lambda^{2n-2}A_{n-1} + \dots + A_0)^{-1}$  exists on the region  $\{\lambda^{1/2} : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta} : |\lambda| \geq r\}$ , which can be contained in an arbitrary acute angle at vertex  $(0, 0)$ ; on the other hand, in the formulation of [28, Theorem 2.1], T.-J. Xiao and J. Liang require the existence of operator  $P_\lambda^{-1}$  for any complex number  $\lambda$  having the modulus greater than or equal to  $r$  and belonging to the obtuse angle  $e^{i\phi}\Sigma_{(\pi/2)+\theta}$ .

In the following slight modification of [33, Example 2], we will focus our attention on the possible applications of Theorem 2 in which  $C$  is not the identity operator on  $X$ ; this is a very important example because of its universality and covering a wide range of concrete applications (here, our attempt is to reconsider and relax, in a certain sense, the very restrictive condition [34, (4.2), p. 168]).

**Example 1.** Suppose that  $0 < \zeta \leq 2$ ,  $\theta \in (-\pi, \pi]$ ,  $r > 0$ ,  $q_n > q_{n-1}$ ,  $\emptyset \neq \Omega \subseteq \mathbb{C}$ ,  $N \in \mathbb{N}$ ,  $A$  is a densely defined closed linear operator in  $X$  satisfying that  $\Omega \subseteq \rho(A)$  and the operator family  $\{(1 + |\lambda|)^{-N}(\lambda - A)^{-1} : \lambda \in \Omega\} \subseteq L(X)$  is equicontinuous (here we can also assume that the operator  $A$  is not densely defined or that the operator family  $\{(1 + |\lambda|)^{-N}(\lambda - A)^{-1}C : \lambda \in \Omega\} \subseteq L(X)$  is equicontinuous, thus providing some applications of Theorem 2 to the abstract degenerate fractional equations involving non-elliptic differential operators with the empty resolvent set; see [39]). Assume, further, that  $P_i(z)$  is a complex polynomial ( $i \in \mathbb{N}_n^0$ ),  $P_n(z) \not\equiv 0$ ,  $\lambda_0 \in \rho(A) \setminus \{z \in \mathbb{C} : P_n(z) = 0\}$ ,  $\text{dist}(\lambda_0, \Omega) > 0$ , as well as that for each  $\lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta}$  with  $|\lambda| \geq r$  all roots of the polynomial

$$z \mapsto \lambda^{q_n} P_n(z) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(z), \quad z \in \mathbb{C}$$

belong to  $\Omega$ . Set  $B := P_n(A)$  and  $A_i := P_i(A)$  ( $i \in \mathbb{N}_{n-1}^0$ ). Then it can be proved that there exist two sufficiently large numbers  $Q' \geq N + 2$ ,  $Q' \in \mathbb{N}$  and  $r' > r$  such that the operator families  $\{(1 + |\lambda|)^{q_n}(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A))^{-1}(\lambda_0 - A)^{-Q'} : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta}, |\lambda| \geq r'\} \subseteq L(X)$  and  $\{(1 + |\lambda|)^{q_n} P_j(A)(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A))^{-1}(\lambda_0 - A)^{-Q'} : \lambda \in e^{i\phi}\Sigma_{(\zeta\pi/2)+\theta}, |\lambda| \geq r'\} \subseteq L(X)$  are both equicontinuous and strongly continuous ( $j \in \mathbb{N}_{n-1}^0$ ). Hence, Theorem 2 can be applied with the regularizing operator  $C \equiv (\lambda_0 - A)^{-Q'}$ .

In the following theorem, we will reconsider the assertion of [8, Theorem 2.3.3] for systems of abstract degenerate differential equations.

**Theorem 5.** *Let  $(X, \|\cdot\|)$  be a complex Banach space and let  $iA_j$ ,  $1 \leq j \leq n$ , be commuting generators of bounded  $C_0$ -groups on  $X$ . Suppose  $\alpha > 0$ ,  $d \in \mathbb{N}$  and  $P_i(x) = \sum_{|\eta| \leq d} P_{\eta,i} x^\eta$  ( $P_{\eta,i} \in \mathbb{C}^{m,m}$ ,  $x \in \mathbb{R}^n$ ,  $i = 1, 2$ ) are two given polynomial matrices. Suppose that for each  $x \in \mathbb{R}^n$  the matrix  $P_2(x)$  is regular. Then there exists a dense subset  $X_{\alpha,m}$  of  $X^m$  such that, for every  $\vec{x} \in X_{\alpha,m}$ , there exists a unique solution (defined in the very*

obvious way) of the following abstract Cauchy problem:

$$(DFP) : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} \vec{u}(t) = \overline{P_2(A)} \mathbf{D}_t^\alpha \vec{u}(t) = \overline{P_1(A)} \vec{u}(t), & t \geq 0, \\ \vec{u}(0) = \vec{x}; & \vec{u}^{(j)}(0) = 0, \quad 1 \leq j \leq \lceil \alpha \rceil - 1. \end{cases}$$

Furthermore, for every  $\vec{x} \in X_{\alpha,m}$ , the mapping  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$ , can be extended to the whole complex plane (the extension of this mapping will be denoted by the same symbol in the sequel), and the following holds:

- (i) the mapping  $z \mapsto \vec{u}(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$  is analytic;
- (ii) the mapping  $z \mapsto \vec{u}(z)$ ,  $z \in \mathbb{C}$  is entire provided that  $\alpha \in \mathbb{N}$ .

*Proof.* Let us recall that  $k = 1 + \lfloor n/2 \rfloor$ . Suppose that  $P_1(x) = [p_{ij;1}(x)]_{1 \leq i,j \leq m}$  and  $P_2(x) = [p_{ij;2}(x)]_{1 \leq i,j \leq m}$  ( $x \in \mathbb{R}^n$ ), and  $d$  is the maximal degree of all non-zero polynomials  $p_{ij;1}(x)$  and  $p_{ij;2}(x)$  ( $1 \leq i, j \leq m$ ). Then  $\sup_{x \in \mathbb{R}^n} |\det(P_2(x))|^{-1} < \infty$  and we can inductively prove that there exist numbers  $M_1 \geq 1$  and  $M_2 \geq 1$  such that for each  $l \in \mathbb{N}_0$  there exist polynomials  $R_{ij;l}(x)$  ( $1 \leq i, j \leq m$ ) of degree  $\leq lmd$  satisfying that

$$\left( P_2(x)^{-1} P_1(x) \right)^l = \frac{[R_{ij;l}(x)]_{1 \leq i,j \leq m}}{(\det(P_2(x)))^l}, \quad x \in \mathbb{R}^n,$$

and that the following holds:

$$\begin{aligned} & \left| D^n \left( \frac{R_{ij;l}(x)}{(\det(P_2(x)))^l} \right) \right| + \left| D^n \left( p_{ij;1}(x) \frac{R_{ij;l}(x)}{(\det(P_2(x)))^l} \right) \right| + \\ & + \left| D^n \left( p_{ij;2}(x) \frac{R_{ij;l}(x)}{(\det(P_2(x)))^l} \right) \right| \leq M_1^l (1 + |x|)^{lmdM_2}, \end{aligned} \tag{9}$$

provided  $l \in \mathbb{N}_0$ ,  $x \in \mathbb{R}^n$ ,  $0 \leq |\eta| \leq k$  and  $1 \leq i, j \leq m$ . It is very simple to prove that there exists a sufficiently large natural number  $k'$  satisfying  $2|k'$  and

$$\lim_{l \rightarrow +\infty} \frac{\left( \Gamma \left( \frac{2M_2(l+1)md+n}{k'd} \right) \right)^{1/2l}}{(\Gamma(\alpha l + 1))^{1/l}} = 0. \tag{10}$$

Let  $a > 0$  be fixed. Set  $C := (e^{-a|x|^{k'd}})(A)$  and  $X_{\alpha,m} := (R(C))^m$ . Then  $C \in L(X)$ ,  $C$  is injective and  $D_\infty(A_1^2 + \dots + A_n^2) \supseteq R(C)$  is dense in  $X$  [37]. Define

$$W_\alpha(z) := \left[ \sum_{l=0}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \left( \frac{R_{ij;l}(x)}{(\det(P_2(x)))^l} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m}, \quad z \in \mathbb{C}. \tag{11}$$

Using (9), (10) and the proof of [8, Theorem 2.3.3], it readily follows that  $W_\alpha(z) \in L(X^m)$  for all  $z \in \mathbb{C}$ , as well as that the expressions

$$\left[ \sum_{l=0}^{\infty} \sum_{v=1}^m \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \left( p_{iv;2}(x) \frac{R_{vj;l+1}(x)}{(\det(P_2(x)))^{l+1}} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m}$$

and

$$\left[ \sum_{l=0}^{\infty} \sum_{v=1}^m \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \left( p_{iv;1}(x) \frac{R_{vj;l}(x)}{(\det(P_2(x)))^l} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m}$$

define the bounded linear operators on  $X^m$  ( $z \in \mathbb{C}$ ). Furthermore, the mapping  $z \mapsto W_\alpha(z)$ ,  $z \in \mathbb{C} \setminus (-\infty, 0]$ , is analytic, and the mapping  $z \mapsto W_\alpha(z)$ ,  $z \in \mathbb{C}$ , is entire provided that  $\alpha \in \mathbb{N}$ . Suppose now  $\vec{x} \in X_{\alpha,m}$ . Then there exists  $x' \in X^m$  such that  $\vec{x} = C_m x'$ , where  $C_m = CI_m$ . Setting  $\vec{u}(z) := W_\alpha(z)x'$ ,  $z \in \mathbb{C}$ , we immediately obtain that (i) and (ii) hold. It is not difficult to prove that  $\mathbf{D}_t^\alpha(t)\vec{u}(t)$  is well-defined, as well as that

$$\mathbf{D}_t^\alpha(t)\vec{u}(t) = \left[ \sum_{l=1}^\infty \frac{t^{\alpha(l-1)}}{\Gamma(\alpha(l-1)+1)} \left( \frac{R_{ij;l}(x)}{(\det(P_2(x)))^l} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m} \vec{x}', \quad t \geq 0,$$

and  $\vec{u}(0) = \vec{x}$ ,  $\vec{u}^{(j)}(0) = 0$ ,  $1 \leq j \leq \lceil \alpha \rceil - 1$ . Since  $\overline{P_1(A)}$  and  $\overline{P_2(A)}$  are closed, we can prove with the help of ( $\triangleright$ ) that  $\vec{u}(t) \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$ ,  $\mathbf{D}_t^\alpha \vec{u}(t) \in D(\overline{P_2(A)})$ , the term  $\mathbf{D}_t^\alpha \overline{P_2(A)} \vec{u}(t)$  is well defined, with

$$\begin{aligned} \overline{P_2(A)} \mathbf{D}_t^\alpha \vec{u}(t) &= \mathbf{D}_t^\alpha \overline{P_2(A)} \vec{u}(t) \\ &= \left[ \sum_{l=0}^\infty \sum_{v=1}^m \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \left( p_{iv;2}(x) \frac{R_{iv;l+1}(x)}{(\det(P_2(x)))^{l+1}} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m} \vec{x}' \end{aligned}$$

and

$$\overline{P_1(A)} \vec{u}(t) = \left[ \sum_{l=0}^\infty \sum_{v=1}^m \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \left( p_{iv;1}(x) \frac{R_{iv;l}(x)}{(\det(P_2(x)))^l} e^{-a|x|^{k'd}} \right) (A) \right]_{1 \leq i,j \leq m} \vec{x}',$$

for any  $t \geq 0$ . Since

$$P_2(x) \frac{[R_{ij;l+1}(x)]_{1 \leq i,j \leq m}}{(\det(P_2(x)))^{l+1}} = P_1(x) \frac{[R_{ij;l}(x)]_{1 \leq i,j \leq m}}{(\det(P_2(x)))^l}, \quad l \in \mathbb{N}_0, \quad x \in \mathbb{R}^n,$$

a simple matricial computation shows that the function  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$ , is a solution of problem (DFP). Now we will prove the uniqueness of solutions of problem (DFP). Let  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$ , be a solution of (DFP) with  $\vec{x} = 0$ . Integrating  $\alpha$ -times (DFP), we get that  $\overline{P_2(A)} \vec{u}(t) = \int_0^t g_\alpha(t-s) \overline{P_1(A)} \vec{u}(s) ds$ ,  $t \geq 0$ . Using this equality, as well as the fact that  $\overline{P_2(A)} W_\alpha(t) - \overline{P_2(A)} C_m = \overline{P_1(A)} (g_\alpha * W_\alpha(\cdot))(t) \in L(X^m)$ ,  $t \geq 0$ , and the proof of [1, Proposition 1.1], we obtain that  $0 = (\overline{W_\alpha} * 0)(t) = (\overline{P_2(A)} C_m * \vec{u})(t)$ ,  $t \geq 0$ , so that it suffices to prove that the operator  $\overline{P_2(A)}$  is injective. Suppose that  $\overline{P_2(A)} \vec{x} = \vec{0}$  for some  $\vec{x} \in X^m$ . By [38, Lemma 1.1(a)], we may assume without loss of generality that  $\vec{x} \in X_0^m$  (cf. ( $\triangleright$ )). It is clear that there exist polynomials  $q_{ij}(x)$  ( $1 \leq i, j \leq m$ ) such that  $P_2(x)^{-1} = (\det(P_2(x)))^{-1} [q_{ij}(x)]_{1 \leq i,j \leq m}$ . Using ( $\triangleright$ ), one can prove that  $[(\det(P_2(x)))(A)I_m][\phi(A)I_m] \vec{x} = [(\phi(x)q_{ij}(x))(A)]_{1 \leq i,j \leq m} P_2(A) \vec{x} = \vec{0}$ ,  $\phi \in \mathcal{S}(\mathbb{R}^n)$ . By [15, Remark 4.4(i)], the operator  $(\det(P_2(x)))(A)$  is injective, whence we may conclude that  $[\phi(A)I_m] \vec{x} = \vec{0}$ ,  $\phi \in \mathcal{S}(\mathbb{R}^n)$ . This, in turn, implies  $\vec{x} = \vec{0}$  and completes the proof of theorem.  $\square$

**Remark 2.** (i) It can be simply proved that for each  $\vec{x} \in X_{\alpha,m}$  the solution  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$ , possesses some expected properties from Definition 1: If  $\alpha \notin \mathbb{N}$ , then the terms  $(\mathbf{D}_t^\alpha)^p \vec{u}(t)$ ,  $\overline{P_1(A)} (\mathbf{D}_t^\alpha)^p \vec{u}(t)$  and  $\overline{P_2(A)} (\mathbf{D}_t^\alpha)^p \vec{u}(t)$  are well-defined and can be extended to the whole complex plane, analytically on the region  $\mathbb{C} \setminus (-\infty, 0]$  and continuously on the region  $\mathbb{C} \setminus (-\infty, 0)$ ; if  $\alpha \in \mathbb{N}$ , then the terms  $(d^{\alpha p}/dt^{\alpha p}) \vec{u}(t)$ ,  $\overline{P_1(A)} (d^{\alpha p}/dt^{\alpha p}) \vec{u}(t)$  and  $\overline{P_2(A)} (d^{\alpha p}/dt^{\alpha p}) \vec{u}(t)$  are well-defined and can be entirely extended to the whole complex

plane ( $p \in \mathbb{N}_0$ ). The assertion of [8, Theorem 2.3.5] can be reformulated in degenerate case, as well.

(ii) If  $m = 1$ ,  $P_1(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ ,  $P_2(x) = \sum_{|\alpha| \leq d} b_\alpha x^\alpha$ ,  $x \in \mathbb{R}^n$  ( $a_\alpha, b_\alpha \in \mathbb{C}$ ),  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$  and  $X$  is a function space on which translations are uniformly bounded and strongly continuous (for example,  $L^p(\mathbb{R}^n)$  with  $p \in [1, \infty)$ ,  $C_0(\mathbb{R}^n)$  or  $BUC(\mathbb{R}^n)$ ; notice also that  $X$  can be consisted of functions defined on some bounded domain [37; 39]), then the obvious choice for  $A_j$  is  $i\partial/\partial x_j$  ( $1 \leq j \leq n$ ). In this case,  $\overline{P_1(A)}$  and  $\overline{P_2(A)}$  are just the operators  $\sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|} (\partial/\partial x)^\alpha$  and  $\sum_{|\alpha| \leq d} b_\alpha i^{|\alpha|} (\partial/\partial x)^\alpha$ , respectively, acting with their maximal distributional domains. Making use of Theorem 5 and a slight modification of the formula appearing on p. 49 of [8], we can prove that for each  $\alpha > 0$  there exists a dense subset  $X_{\alpha,1}$  of  $L^p(\mathbb{R}^n)$  such that the abstract Cauchy problem:

$$\begin{aligned} \sum_{|\alpha| \leq d} b_\alpha i^{|\alpha|} (\partial/\partial x)^\alpha \mathbf{D}_t^\alpha u(t, x) &= \mathbf{D}_t^\alpha \sum_{|\alpha| \leq d} b_\alpha i^{|\alpha|} (\partial/\partial x)^\alpha u(t, x) = \\ &= \sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|} (\partial/\partial x)^\alpha u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n; \\ \frac{\partial^l}{\partial t^l} u(t, x)|_{t=0} &= f_l(x), \quad x \in \mathbb{R}^n, \quad l = 0, 1, \dots, [\alpha] - 1, \end{aligned}$$

has a unique solution (obeying the properties clarified in the part (i) of this remark) provided  $f_l(\cdot) \in X_{\alpha,1}$ ,  $l = 0, 1, \dots, [\alpha] - 1$ . A similar assertion can be formulated in  $X_l$ -type spaces [34]; we can also move to the spaces  $L^\infty(\mathbb{R}^n)$ ,  $C_b(\mathbb{R}^n)$  or  $C^\sigma(\mathbb{R}^n)$  ( $0 < \sigma < 1$ ) by using distributional techniques, but then we cannot expect the density of the corresponding subspace  $X_{\alpha,1}$ .

(iii) In [14; 15], we have recently considered the  $C$ -wellposedness of the abstract degenerate Cauchy problem

$$(\text{DFP}) : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} u(t) = \overline{P_2(A)} \mathbf{D}_t^\alpha u(t) = \overline{P_1(A)} u(t), & t \geq 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, & 1 \leq j \leq [\alpha] - 1, \end{cases}$$

where  $0 < \alpha \leq 2$ ,  $P_1(x)$  and  $P_2(x)$  are complex polynomials,  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$ ,  $iA_j$ ,  $1 \leq j \leq n$ , are commuting generators of bounded  $C_0$ -groups on a Banach space  $X$ , thus continuing the research studies of T.-J. Xiao and J. Liang [19; 22]. Denote  $\Omega(\omega) = \{\lambda^2 : \text{Re } \lambda > \omega\}$ , if  $\omega > 0$ , and  $\Omega(\omega) = \mathbb{C} \setminus (-\infty, -\omega^2]$ , if  $\omega \leq 0$ . The basic assumption in [14; 15] was that

$$\sup_{x \in \mathbb{R}^n} \text{Re} \left( \left( \frac{P_1(x)}{P_2(x)} \right)^{1/\alpha} \right) \leq \omega,$$

provided  $0 < \alpha < 2$ , and  $P_1(x)/P_2(x) \notin \Omega(\omega)$ ,  $x \in \mathbb{R}^n$ , provided  $\alpha = 2$ . Observe that our results from the part (i) of this remark can be applied in the analysis of problem (DFP) in the general case  $\alpha > 0$ , and also in the case that  $0 < \alpha \leq 2$  and the above-mentioned requirements are not satisfied.

## 2.1. Final conclusions and remarks

We feel duty bound to say that Theorem 5 and the conclusions from the parts of (ii) and (iii) of Remark 2 cannot be applied in the analysis of  $L^p$ -wellposedness of a great

number of very important degenerate equations of mathematical physics, like (cf. the monograph by G.V. Demidenko and S.V. Uspenskii [20] for further information):

- (a) (the Rossby wave equation, 1939)

$$\Delta u_t + \beta u_y = 0 \quad (n = 2), \quad u(0, x, y) = u_0(x, y);$$

- (b) (the Sobolev equation, 1940)

$$\begin{aligned} \Delta u_{tt} + \omega^2 u_{zz} &= 0 \quad (n = 3), \\ u(0, x, y, z) &= u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z), \end{aligned}$$

here  $\omega/2$  is the angular velocity;

- (c) (the internal wave equation in the Boussinesq approximation, 1903)

$$\begin{aligned} \Delta u_{tt} + N^2(u_{xx} + u_{yy}) &= 0 \quad (n = 3), \\ u(0, x, y, z) &= u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z); \end{aligned}$$

- (d) (the gravity-gyroscopic wave equation, cf. [42])

$$\begin{aligned} \Delta u_{tt} + N^2(u_{xx} + u_{yy}) + \omega^2 u_{zz} &= 0 \quad (n = 3), \\ u(0, x, y, z) &= u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z); \end{aligned}$$

- (e) (small amplitude oscillations of a rotating viscous fluid)

$$\begin{aligned} \Delta u_{tt} - 2\nu \Delta^2 u_t + \nu^2 \Delta^3 u + \omega^2 u_{zz} &= 0 \quad (n = 3), \\ u(0, x, y, z) &= u_0(x, y, z), \quad u_t(0, x, y, z) = u_1(x, y, z). \end{aligned}$$

Here  $\omega/2$  is the angular velocity and  $\nu > 0$  is the viscosity coefficient.

Before including some details on the existence and uniqueness of entire solutions of the equations (a)–(e) in  $L^p$  spaces, we need to explain how one can reformulate the assertion of Theorem 5 in the case that there exist a vector  $x_0 \in \mathbb{R}^n$  and a non-empty compact subset  $K$  of  $\mathbb{R}^n$  such that the matrix  $P_2(x_0)$  is singular and  $\{x \in \mathbb{R}^n : \det(P_2(x)) = 0\} \subseteq K$ ; the analysis of some fractional analogons of (a)–(e) can be carry out similarly and is therefore omitted. Denote by  $\mathcal{A}$  the class consisting of those  $C^\infty(\mathbb{R}^n)$ -functions  $\phi(\cdot)$  satisfying that there exist two open relatively compact neighborhoods  $\Omega$  and  $\Omega'$  of  $K$  in  $\mathbb{R}^n$  such that  $\phi(x) = 0$  for all  $x \in \Omega$  and  $\phi(x) = 1$  for all  $x \in \mathbb{R}^n \setminus \Omega'$ . Since the estimate (9) holds for all  $x \in \mathbb{R}^n \setminus \Omega$ , we can define for each  $z \in \mathbb{C}$  the matricial operator  $W_\alpha(z)$  (cf. the proof of Theorem 5) by replacing the function  $e^{-a|x|^{k'd}}$  in (11) with the function  $\phi(x)e^{-a|x|^{k'd}}$ . Setting  $C_\phi := (\phi(x)e^{-a|x|^{k'd}})(A)$  for  $\phi \in \mathcal{A}$  (then we do not know any longer whether the set  $\bigcup_{\phi \in \mathcal{A}} R(C_\phi)$  is dense in  $X$ , and we cannot clarify whether the operator  $C_\phi$  is injective or not) and  $X'_{\alpha, m} := (\bigcup_{\phi \in \mathcal{A}} R(C_\phi))^m$ , and assuming additionally the injectivity of matricial operator  $\overline{P_2(A)}$  on  $X^m$ , then for each  $\vec{x} \in X'_{\alpha, m}$  there exists a unique solution  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$ , of the abstract Cauchy problem (DFP), which can be extended to the whole complex plane, and (i), (ii) from the formulation of Theorem 5 continues to hold. Rewriting any of the equations (a)–(e) in the matricial form, and using Lemma 2, we obtain that there exists a non-trivial subspace  $X'_{1,1}$  of  $L^p(\mathbb{R}^2)$ , resp.  $X'_{1,2}$  of  $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ , such that the equation (a), resp. any of the equations (b)–(e), has a unique entire solution provided  $u_0(x, y) \in$

$X'_{1,1}$ , resp.  $(u_0(x, y, z), u_1(x, y, z)) \in X'_{1,2}$  (similar conclusions can be established for the wellposedness of the equations [43, (5), (6); Section 4] in  $L^p(\Omega)$ , with  $\tau > 0$ ,  $\lambda_2 > 0$  and  $\lambda_3 > 0$  and  $\emptyset \neq \Omega \subseteq \mathbb{R}^n$  being an open bounded domain with smooth boundary, which are important in the study of fluctuations of thermoelastic plates and non-stationary processes in thermal fields; the uniqueness of entire solutions of the equation [43, (5)], resp. [43, (6)], in the case that  $\lambda_2 < 0$ , resp.  $\lambda_3 < 0$ , cannot be proved here by using the injectivity of associated polynomial matrix operator  $\overline{P_2(A)}$ , and we will only refer the reader to the assertions of [43, Theorem 4] and [14, Theorem 2.2] for further information in this direction). It should be finally noted that we do not know, in the present situation, whether the subspace  $X'_{1,1}$ , resp.  $X'_{1,2}$ , of initial values  $\vec{x} = u_0$ , resp.  $\vec{x} = (u_0, u_1)$ , for which there exists a unique entire solution  $t \mapsto \vec{u}(t)$ ,  $t \geq 0$  of the equation (a), resp. any of the equations (b)–(e), can be chosen to be dense in  $L^p(\mathbb{R}^2)$ , resp.  $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ . This is an open problem we would like to propose to our readers.

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## **ЦЕЛЫЕ И АНАЛИТИЧЕСКИЕ РЕШЕНИЯ АБСТРАКТНЫХ ВЫРОЖДЕННЫХ ДРОБНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ<sup>1</sup>**

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Исследуются целые и аналитические решения абстрактных вырожденных (с несколькими слагаемыми) дробных дифференциальных уравнений с дробной производной Капуто по времени. Мы также анализируем системы таких уравнений и предоставляем несколько иллюстративных примеров, демонстрирующих полезность полученных теоретических результатов.

**Ключевые слова:** *абстрактное вырожденное дифференциальное уравнение, интегро-дифференциальное уравнение Вольтерры, дробное дифференциальное уравнение, целая функция, аналитическая функция, корректность.*

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