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CORRESPONDENCE ANALYSIS FOR LOGIC OF RATIONAL AGENT

Y. I. Petrukhin

Lomonosov Moscow State University, Moscow, Russia
yaroslav.petrukhin@mail.ru

In this paper, we examine Kubyshkina & Zaitsev's Logic of Rational Agent (LRA) from a proof-theoretic point of view. We present three natural deduction systems for LRA which differ from Kubyshkina & Zaitsev's axiomatization of LRA. Moreover, we introduce a general method for axiomatizing LRA's unary and binary truth-functional extensions via natural deduction systems. This method is Kooi & Tamminga's correspondence analysis which we adapt for LRA.

Keywords: *many-valued logics, generalized truth values, correspondence analisys, natural deduction systems.*

Introduction

Kubyshkina & Zaitsev's Logic of Rational Agent (LRA) [1] is both one of so called logics of generalized classical truth values and one of epistemic logics. The generalization technique of classical truth values allows the authors "to introduce a system, where the epistemic operator for knowledge (K_a-operator) does not appear, but the fact of knowing or not knowing some truths (or the falsity of some statement) can be defined truth-functionally" [1, p. 2]. The avoiding of the use of K_a-operator allows to escape so called Church — Fitch's paradox (knowability paradox): $p \to \mathsf{K}_a p$, i.e. if it holds that p then agent a knows that p. This paradox arises in some epistemic systems. See [1] for the review of solutions of this paradox. Recall that one of solutions is reasoning with LRA which is based on generalized classical truth values.

In [2], Dunn suggested the idea of generalization of classical truth values. He considered the subsets of the set $\{\mathbf{T}, \mathbf{F}\}$ of classical "true" and "false" as independent truth values. As a consequence, he obtained a very simple and intuitive four-valued semantics for FDE [3; 4] with the set $\{\emptyset, \{\mathbf{T}\}, \{\mathbf{F}\}, \{\mathbf{T}, \mathbf{F}\}\}$ of truth values. Belnap [5; 6] interpreted these values as follows: $\emptyset$ = "none", $\{\mathbf{T}\}$ = "true", $\{\mathbf{F}\}$ = "false", and $\{\mathbf{T}, \mathbf{F}\}$ = "both". Later on the generalization technique of classical truth values have been developing by Shramko, Dunn & Takenaka [7], Shramko & Wansing [8], Zaitsev & Grigoriev [9; 10], Zaitsev & Shramko [11], Zaitsev [12], Grigoriev [13] and Wintein & Muskens [14].

In case of LRA, the generalization technique of classical truth values works as follows: consider the set $\{\mathbf{T}, \mathbf{F}\}$ of truth values "true" and "false" and the set $\{\mathbf{1}, \mathbf{0}\}$ of truth values "known" and "unknown". These sets deal with ontological truth and falsehood and epistemic states of the agent, respectively. Then $\{\mathbf{T}, \mathbf{F}\}$ and $\{\mathbf{1}, \mathbf{0}\}$ are multiplied. As a result, the set $\{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$ of truth values arises. The set of designated values is $\{\mathbf{T1}\}$.

The propositional formula of LRA ($\mathcal{L}$-formula) is defined in a standard way from the propositional variables $p, q, r, p_1, \ldots$, unary operators $\neg$, $\sim$ and binary operators $\wedge$, $\vee$.

A logical matrix for LRA is $\mathfrak{M}_{LRA} = \langle\{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}, f_\neg, f_\sim, f_\wedge, f_\vee, \{\mathbf{T1}\}\rangle$. Functions $f_\neg, f_\sim, f_\wedge, f_\vee$ are defined as follows:

φ	$f_\neg$	$f_\sim$
T1	**F1**	**T0**
T0	**F0**	**T1**
F1	**T1**	**F0**
F0	**T0**	**F1**

$f_\wedge$	**T1**	**T0**	**F1**	**F0**
T1	**T1**	**T0**	**F1**	**F0**
T0	**T0**	**T0**	**F1**	**F0**
F1	**F1**	**F1**	**F1**	**F1**
F0	**F0**	**F0**	**F1**	**F0**

$f_\vee$	**T1**	**T0**	**F1**	**F0**
T1	**T1**	**T1**	**T1**	**T1**
T0	**T1**	**T0**	**T0**	**T0**
F1	**T1**	**T0**	**F1**	**F0**
F0	**T1**	**T0**	**F0**	**F0**

The notion of valuation of formula A in $\mathfrak{M}_{LRA}$ is defined in a standard way. Let Γ be an arbitrary set of $\mathcal{L}$-formulas and A be an arbitrary $\mathcal{L}$-formula. Then $\Gamma \models_{LRA} A$ iff for each valuation v if $v(B) = \mathbf{T1}$ (for each $B \in \Gamma$) then $v(A) = \mathbf{T1}$. If $\Gamma = \varnothing$ then A is said to be a tautology.

Note that $\{\neg,\wedge,\vee\}$-fragment of LRA was first studied in Kubyshkina's abstract [15] and Zaitsev's paper [16]. Connectives $\neg$ and $\sim$, respectively, are an ontological and epistemic negations, i.e. $\neg$ changes '**T**' and '**F**' (ontological parts of truth values) and $\sim$ changes '**1**' and '**0**' (epistemic parts of truth values).

The purpose of this paper is to present correspondence analysis for LRA. Kooi & Tamminga have first described this framework in their paper [17] where they considered unary and binary extensions of three-valued paraconsistent logic LP (Logic of Paradox) [18–20]. Later on Tamminga [21] applied this technique to strong three-valued logic K_3 [22]. In [23], correspondence analysis was extended to four-valued relevant logic FDE [2–6].

Consider a logic $LRA_{\star\circ}$ with $\mathcal{L}_{\star\circ}$-formulas constructed in a standard way from the propositional variables and unary operators $\neg$, $\sim$, $\star_1, \ldots \star_n$ and binary operators $\vee$, $\wedge$, $\circ_1, \ldots, \circ_m$.

Let $\mathfrak{M}_{LRA_{\star\circ}} = \langle\{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}, f_\neg, f_\sim, f_\wedge, f_\vee, f_{\star_1}, \ldots, f_{\star_n}, f_{\circ_1}, \ldots, f_{\circ_m}, \{\mathbf{T1}\}\rangle$ be a logical matrix for $LRA_{\star\circ}$. The notion of valuation of formula A in $\mathfrak{M}_{LRA_{\star\circ}}$ is defined in a standard way. Let Γ be an arbitrary set of $\mathcal{L}_{\star\circ}$-formulas and A be an arbitrary $\mathcal{L}_{\star\circ}$-formula. Then $\Gamma \models_{LRA_{\star\circ}} A$ iff for each valuation v if $v(B) = \mathbf{T1}$ (for each $B \in \Gamma$) then $v(A) = \mathbf{T1}$. If $\Gamma = \varnothing$ then A is said to be a tautology.

So the first step of correspondence analysis is a characterization of all 16 possible equalities of the form $\star a = b$ and all 64 possible equalities of the form $a \circ b = c$ by inference schemes. On the second step we consider these inference schemes as *inference rules*. Using them and a natural deduction system for LRA, we define a class of natural deduction systems for LRA's extensions. On the third step we show soundness and completeness of these calculi with respect to their semantics.

1. Correspondence analysis for LRA

In this section, we introduce inference schemes for operators $\circ$ and $\star$. If φ is $\mathcal{L}_{\star\circ}$-formula or $\varphi \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$, then we introduce the notations

$$\varphi^{\mathbf{T1}} = \varphi, \quad \varphi^{\mathbf{T0}} = \sim\varphi, \quad \varphi^{\mathbf{F1}} = \neg\varphi, \quad \varphi^{\mathbf{F0}} = \sim\neg\varphi. \tag{1}$$

By (1) for every $a \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$

$$a^a = \mathbf{T1} \tag{2}$$

and for every $\mathcal{L}_{\star\circ}$-formula φ and for every valuation v

$$v(\varphi^a) = \mathbf{T1} \Longleftrightarrow v(\varphi) = a. \tag{3}$$

Theorem 1. *Let* $a, b, c \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$. *Then*

$$a \circ b = c \iff [\varphi^a \wedge \psi^b \models (\varphi \circ \psi)^c \quad \textit{for any } \mathcal{L}_{\star\circ}\textit{-formulas } \varphi, \psi]. \tag{4}$$

Proof. ($\Rightarrow$) Let $a \circ b = c$. Given any valuation v suppose $v(\varphi^a) = \mathbf{T1}$, $v(\psi^b) = \mathbf{T1}$. By (3) $v(\varphi) = a$, $v(\psi) = b$. Hence $v((\varphi \circ \psi)^c) = (a \circ b)^c = c^c$. By (2) $v((\varphi \circ \psi)^c) = \mathbf{T1}$.

($\Leftarrow$) Let $\varphi^a \wedge \psi^b \models (\varphi \circ \psi)^c$ for any $\mathcal{L}_{\star\circ}$-formulas φ, ψ. Put $\varphi = p$, $\psi = q$ (p, q are propositional variables). Choose $v(p) = a$, $v(q) = b$. Then by (2) $v(p^a) = v(q^b) = \mathbf{T1}$. Applying $p^a \wedge q^b \models (p \circ q)^c$ gives $v((p \circ q)^c) = \mathbf{T1}$, hence by (3) $a \circ b = c$. □

Theorem 2. *Let* $a, b \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$. *Then*

$$\star a = b \iff [\varphi^a \models (\star\varphi)^b \quad \textit{for any } \mathcal{L}_{\star\circ}\textit{-formula } \varphi]. \tag{5}$$

Proof. Adapt the proof of Theorem 1. □

Let us substitute $a \circ b$ to the place of c in (4). Also let us substitute $\star a$ to the place of b in (5). We get

Theorem 3. *For any* $a, b \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$ *and for any* $\mathcal{L}_{\star\circ}$*-formulas* φ, ψ

$$\varphi^a \wedge \psi^b \models (\varphi \circ \psi)^{a \circ b}, \qquad \varphi^a \models (\star\varphi)^{\star a}. \tag{6}$$

Due to (4), (5) and (6) applying (1) we obtain characteristics in the spirit of Kooi & Tamminga [17; 21] for 80 equalities of the form $a \circ b = c$ and $\star a = b$. For example, combining $a = \mathbf{T0}$, $b = \mathbf{F1}$, $c = \mathbf{F0}$ with (4), (1) yields

$$\mathbf{T0} \circ \mathbf{F1} = \mathbf{F0} \iff [\sim \varphi \wedge \neg\psi \models \sim\neg(\varphi \circ \psi) \textit{ for any } \mathcal{L}_{\star\circ}\textit{-formulas } \varphi, \psi]. \tag{7}$$

It is easy to prove the following analog of (6):

$$\varphi^a \wedge \psi^b \models (\varphi \wedge \psi)^{a \wedge b}, \quad \varphi^a \wedge \psi^b \models (\varphi \vee \psi)^{a \vee b}, \quad \varphi^a \models (\neg\varphi)^{\neg a}, \quad \varphi^a \models (\sim \varphi)^{\sim a}. \tag{8}$$

2. Natural deduction systems

A natural deduction system $\mathfrak{ND}^1_{LRA}$ for LRA is as follows:

- Axiom:

$$(EM)\ \varphi \vee \neg\varphi \vee \sim\varphi \vee \sim\neg\varphi.$$

- Rules for negations:

$$(EFQ_1)\ \frac{\varphi \wedge \neg\varphi}{\psi}, \quad (EFQ_2)\ \frac{\varphi \wedge \sim\varphi}{\psi}, \quad (EFQ_3)\ \frac{\varphi \wedge \sim\neg\varphi}{\psi},$$

$$(EFQ_4)\ \frac{\neg\varphi \wedge \sim\varphi}{\psi}, \quad (EFQ_5)\ \frac{\neg\varphi \wedge \sim\neg\varphi}{\psi}, \quad (EFQ_6)\ \frac{\sim\varphi \wedge \sim\neg\varphi}{\psi},$$

$$(\neg\neg I)\ \frac{\varphi}{\neg\neg\varphi}, \quad (\sim\sim I)\ \frac{\varphi}{\sim\sim\varphi}, \quad (\sim\neg I)\ \frac{\neg\sim\varphi}{\sim\neg\varphi},$$

$$(\neg\sim I)\ \frac{\sim\neg\varphi}{\neg\sim\varphi}, \quad (\sim\neg\neg I)\ \frac{\sim\varphi}{\sim\neg\neg\varphi}, \quad (\neg\sim\sim I)\ \frac{\neg\varphi}{\neg\sim\sim\varphi},$$

- Positive fragment:

$$(\wedge I)\ \frac{\varphi \quad \psi}{\varphi \wedge \psi}, \quad (\vee I_1)\ \frac{\varphi}{\varphi \vee \psi}, \quad (\vee I_2)\ \frac{\psi}{\varphi \vee \psi},$$

$$(\vee E)\quad \frac{\varphi_1 \vee \ldots \vee \varphi_n \quad \begin{matrix}[\varphi_1]\\ \chi\end{matrix} \quad \begin{matrix}\ldots\\ \ldots\end{matrix} \quad \begin{matrix}[\varphi_n]\\ \chi\end{matrix}}{\chi},$$

where $n \geqslant 2$ and $[\omega]$ means that the assumption ω is discharged.

• Rules for negations of conjunction and disjunction:

$$(\neg \vee I)\ \frac{\neg\varphi \wedge \neg\psi}{\neg(\varphi \vee \psi)}, \quad (\neg \wedge I)\ \frac{\neg\varphi \vee \neg\psi}{\neg(\varphi \wedge \psi)}, \quad (\sim\neg \vee I)\ \frac{\sim(\neg\varphi \wedge \neg\psi)}{\sim\neg(\varphi \vee \psi)},$$

$$(\sim\wedge I)\ \frac{(\sim\varphi \wedge \psi) \vee (\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \sim\psi)}{\sim(\varphi \wedge \psi)},$$

$$(\sim\vee I)\ \frac{(\sim\varphi \wedge \sim\psi) \vee (\sim\neg\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \sim\neg\psi) \vee (\neg\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \neg\psi)}{\sim(\varphi \vee \psi)},$$

$$(\sim\neg \wedge I)\ \frac{(\sim\neg\varphi \wedge \psi) \vee (\sim\neg\varphi \wedge \sim\psi) \vee (\sim\neg\varphi \wedge \sim\neg\psi) \vee (\sim\varphi \wedge \sim\neg\psi) \vee (\varphi \wedge \sim\neg\psi)}{\sim\neg(\varphi \wedge \psi)}.$$

The notion of a derivation of φ from Γ in $\mathfrak{ND}^1_{LRA}$ and other natural deduction systems described in this paper is defined in a tree-format (Gentzen-style) in a standard way. We introduce an example of derivation in $\mathfrak{ND}^1_{LRA}$ in Figure 1.

$$\dfrac{p \vee \neg p \vee \sim p \vee \sim\neg p \qquad [p] \qquad \dfrac{\dfrac{[\neg p] \qquad \neg\neg p}{\neg p \wedge \neg\neg p}\,(\wedge I)}{p}\,(EFQ_1) \qquad \dfrac{\dfrac{\neg\neg p \qquad \dfrac{[\sim p]}{\sim\neg\neg p}\,(\sim\neg\neg I)}{\neg\neg p \wedge \sim\neg\neg p}\,(\wedge I)}{p}\,(EFQ_2) \qquad \dfrac{\dfrac{\neg\neg p \qquad [\sim\neg p]}{\neg\neg p \wedge \sim\neg p}\,(\wedge I)}{p}\,(EFQ_4)}{p}\,(EM), (\vee E)$$

Fig. 1. A derivation of p from $\neg\neg p$

Proposition 1. *The following rules are derivable in* $\mathfrak{ND}^1_{LRA}$:

$$(\neg\neg E)\ \frac{\neg\neg\varphi}{\varphi}, \quad (\sim\sim E)\ \frac{\sim\sim\varphi}{\varphi}, \quad (\sim\neg\neg E)\ \frac{\sim\neg\neg\varphi}{\sim\varphi}, \quad (\neg\sim\sim E)\ \frac{\neg\sim\sim\varphi}{\neg\varphi},$$

$$(\wedge E_1)\ \frac{\varphi \wedge \psi}{\varphi}, \quad (\wedge E_2)\ \frac{\varphi \wedge \psi}{\psi}, \quad (\neg \vee E)\ \frac{\neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi}, \quad (\neg \wedge E)\ \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \vee \neg\psi},$$

$$(\sim\neg \vee E)\ \frac{\sim\neg(\varphi \vee \psi)}{\sim(\neg\varphi \wedge \neg\psi)}, \quad (\sim\wedge E)\ \frac{\sim(\varphi \wedge \psi)}{(\sim\varphi \wedge \psi) \vee (\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \sim\psi)},$$

$$(\sim\vee E)\ \frac{\sim(\varphi \vee \psi)}{(\sim\varphi \wedge \sim\psi) \vee (\sim\neg\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \sim\neg\psi) \vee (\neg\varphi \wedge \sim\psi) \vee (\sim\varphi \wedge \neg\psi)},$$

$$(\sim\neg \wedge E)\ \frac{\sim\neg(\varphi \wedge \psi)}{(\sim\neg\varphi \wedge \psi) \vee (\sim\neg\varphi \wedge \sim\psi) \vee (\sim\neg\varphi \wedge \sim\neg\psi) \vee (\sim\varphi \wedge \sim\neg\psi) \vee (\varphi \wedge \sim\neg\psi)}.$$

Proposition 2. *Let* $R_{LRA2} := \{(EM), (EFQ_1), (EFQ_2), (EFQ_3), (EFQ_4), (EFQ_5), (EFQ_6), (\neg\neg E), (\sim\sim E), (\sim\neg I), (\neg\sim I), (\sim\neg\neg E), (\neg\sim\sim E), (\wedge I), (\wedge E_1), (\wedge E_2), (\vee E), (\neg \vee E), (\neg \wedge E), (\sim\neg \vee E), (\sim\wedge E), (\sim\vee E), (\sim\neg \wedge E)\ \}$ *be a set of inference rules for natural deduction system* $\mathfrak{ND}^2_{LRA}$*. Then for any set of* $\mathcal{L}$*-formulas* Γ *and for any* $\mathcal{L}$*-formula* φ

$$\Gamma \vdash \varphi \quad in \quad \mathfrak{ND}^1_{LRA} \Longleftrightarrow \Gamma \vdash \varphi \quad in \quad \mathfrak{ND}^2_{LRA}.$$

The Priest's results [20] show that $R_{K_3} := \{(EFQ_1), (\neg\neg I), (\neg\neg E), (\wedge I), (\wedge E_1), (\wedge E_2), (\vee I_1), (\vee I_2), (\vee E), (\neg\vee I), (\neg\vee E), (\neg\wedge I), (\neg\wedge E)\}$ is a set of inference rules for strong Kleene's logic K_3 [22]. Therefore K_3 is a fragment of LRA.

Proposition 3. *Let* $R_{LRA} := \{(EM), (EFQ_1), (EFQ_2), (EFQ_3), (EFQ_4), (EFQ_5), (EFQ_6), (\neg\neg I), (\neg\neg E), (\sim\sim I), (\sim\sim E), (\sim\neg I), (\neg\sim I), (\sim\neg\neg I), (\sim\neg\neg E), (\neg\sim\sim I), (\neg\sim\sim E), (\wedge I), (\wedge E_1), (\wedge E_2), (\vee I_1), (\vee I_2), (\vee E), (\neg\vee I), (\neg\vee E), (\neg\wedge I), (\neg\wedge E), (\sim\neg\vee I), (\sim\neg\vee E), (\sim\wedge I), (\sim\wedge E), (\sim\vee I), (\sim\vee E), (\sim\neg\wedge I), (\sim\neg\wedge E)\}$ *be a set of inference rules for a natural deduction system* $\mathfrak{ND}_{LRA}$. *Then* $\Gamma\vdash\varphi$ *in* $\mathfrak{ND}^1_{LRA} \Longleftrightarrow \Gamma\vdash\varphi$ *in* $\mathfrak{ND}^2_{LRA} \Longleftrightarrow \Gamma\vdash\varphi$ *in* $\mathfrak{ND}_{LRA}$.

Although all these natural deduction systems are deductively equivalent, hereafter we will work with $\mathfrak{ND}_{LRA}$, because it seems to be the most convenient one.

A natural deduction system $\mathfrak{ND}_{LRA_{\star\circ}}$ is an extension of $\mathfrak{ND}_{LRA}$ by inference rules based on Theorem 3: for any $a, b \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$ we add the rules (cf. (6))

$$R_\circ(a,b)\frac{\varphi^a\wedge\psi^b}{(\varphi\circ\psi)^{a\circ b}}, \qquad R_\star(a)\frac{\varphi^a}{(\star\varphi)^{\star a}}. \tag{9}$$

For example, if $\mathbf{T0}\circ\mathbf{F1} = \mathbf{F0}$ (see (7)) and $\star\mathbf{T1} = \mathbf{T0}$, then $\mathfrak{ND}_{LRA}$ is extended by the rules $R_\circ(\mathbf{T0},\mathbf{F1})$ and $R_\star(\mathbf{T1})$:

$$R_\circ(\mathbf{T0},\mathbf{F1})\frac{\sim\varphi\wedge\neg\psi}{\sim\neg(\varphi\circ\psi)}, \qquad R_\star(\mathbf{T1})\frac{\varphi}{\sim(\star\varphi)}.$$

Proposition 4. *For any* $a, b \in \{\mathbf{T1}, \mathbf{T0}, \mathbf{F1}, \mathbf{F0}\}$ *the inference rules* (*cf.* (8))

$$R_\wedge(a,b)\frac{\varphi^a\wedge\psi^b}{(\varphi\wedge\psi)^{a\wedge b}}, \quad R_\vee(a,b)\frac{\varphi^a\wedge\psi^b}{(\varphi\vee\psi)^{a\vee b}}, \quad R_\neg(a)\frac{\varphi^a}{(\neg\varphi)^{\neg a}}, \quad R_\sim(a)\frac{\varphi^a}{(\sim\varphi)^{\sim a}} \tag{10}$$

either are rules of $\mathfrak{ND}_{LRA_{\star\circ}}$, *or derivable in* $\mathfrak{ND}_{LRA_{\star\circ}}$.

3. Soundness and completeness of $\mathfrak{ND}_{LRA_{\star\circ}}$

Soundness follows by a simple routine check.

Theorem 4. (*Soundness*). *For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formula* φ $\Gamma\vdash\varphi \Longrightarrow \Gamma\models\varphi$.

Completeness proof proceeds by Henkin's method [24]. We follow the notational conventions of [17; 21].

Definition 1. *For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formulas* φ *and* ψ Γ *is a nontrivial prime theory, if the following conditions hold*:

(Γ1) $\Gamma \neq Form_{\star\circ}$ *where* $Form_{\star\circ}$ *is a set of all* $\mathcal{L}_{\star\circ}$*-formulas* (*non-triviality*);

(Γ2) $\Gamma\vdash\varphi \Longleftrightarrow \varphi\in\Gamma$ (*closure property of* Γ);

(Γ3) $\varphi_1\vee\ldots\vee\varphi_n\in\Gamma \Longrightarrow (\varphi_1\in\Gamma$ *or* $\ldots$ *or* $\varphi_n\in\Gamma)$ *where* $n\geqslant 2$ (*primeness*).

Definition 2. *For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formula* φ $e(\varphi,\Gamma)$ *is a canonic valuation, if the following conditions hold*:

$$e(\varphi,\Gamma) = \begin{cases} \mathbf{T1} & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \mathbf{T0} & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \mathbf{F1} & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \mathbf{F0} & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 1 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 2 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \emptyset 3 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 4 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \emptyset 5 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 6 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \emptyset 7 & \Longleftrightarrow \quad \varphi \in \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 8 & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 9 & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma; \\ \emptyset 10 & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \in \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 11 & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \in \Gamma, \text{ and } {\sim}\neg\varphi \in \Gamma; \\ \emptyset 12 & \Longleftrightarrow \quad \varphi \notin \Gamma,\ \neg\varphi \notin \Gamma,\ {\sim}\varphi \notin \Gamma, \text{ and } {\sim}\neg\varphi \notin \Gamma. \end{cases}$$

Lemma 1. *For any set of $\mathcal{L}_{\star\circ}$-formulas Γ and for any $\mathcal{L}_{\star\circ}$-formulas φ and ψ*

(1) $e(\varphi,\Gamma) \neq \emptyset i$ *where* $1 \leq i \leq 12$;

(2) $e(\varphi,\Gamma) \circ e(\psi,\Gamma) = e(\varphi \circ \psi,\Gamma)$;

(3) $\star e(\varphi,\Gamma) = e(\star\varphi,\Gamma)$;

(4) $\neg e(\varphi,\Gamma) = e(\neg\varphi,\Gamma)$;

(5) $\sim e(\varphi,\Gamma) = e({\sim}\varphi,\Gamma)$;

(6) $e(\varphi,\Gamma) \vee e(\psi,\Gamma) = e(\varphi \vee \psi,\Gamma)$;

(7) $e(\varphi,\Gamma) \wedge e(\psi,\Gamma) = e(\varphi \wedge \psi,\Gamma)$.

Proof. (1) Suppose $\varphi \in \Gamma$ and $\neg\varphi \in \Gamma$. Then by $(\wedge I)$ and (EFQ_1) $\Gamma = Form_{\star\circ}$, contrary to ($\Gamma$1). Therefore, $e(\varphi,\Gamma) \neq \emptyset i$ where $1 \leq i \leq 4$.

Repeating the previous arguments with using the rules (EFQ_2)–(EFQ_6) instead of (EFQ_1) leads to $e(\varphi,\Gamma) \neq \emptyset i$ where $5 \leq i \leq 11$.

It remains to prove that $e(\varphi,\Gamma) \neq \emptyset 12$. Suppose $\varphi \notin \Gamma$, $\neg\varphi \notin \Gamma$, $\sim\varphi \notin \Gamma$, and ${\sim}\neg\varphi \notin \Gamma$. However, by ($\Gamma$3) and (EM) $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ or $\sim\varphi \in \Gamma$ or $\sim\neg\varphi \in \Gamma$, a contradiction. Therefore, $e(\varphi,\Gamma) \neq \emptyset 12$.

(2) From the preceding part of the Lemma and Definition 2 we deduce for any $a \in \{\mathbf{T1},\mathbf{T0},\mathbf{F1},\mathbf{F0}\}$ and for any $\mathcal{L}_{\star\circ}$-formula φ (see also (1))

$$e(\varphi,\Gamma) = a \Longleftrightarrow \varphi^a \in \Gamma, \tag{11}$$

$$\varphi^{e(\varphi,\Gamma)} \in \Gamma. \tag{12}$$

By (12) and $R_\circ(e(\varphi,\Gamma), e(\psi,\Gamma))$ (see (9)) we obtain $\Gamma \vdash (\varphi \circ \psi)^{e(\varphi,\Gamma)\circ e(\psi,\Gamma)}$. Hence, by the closure property of Γ we have $(\varphi \circ \psi)^{e(\varphi,\Gamma)\circ e(\psi,\Gamma)} \in \Gamma$, and by (11) we conclude $e(\varphi,\Gamma) \circ e(\psi,\Gamma) = e(\varphi \circ \psi,\Gamma)$.

(3) By (12) and $R_\star(e(\varphi,\Gamma))$ (see (9)) we get $\Gamma \vdash (\star\varphi)^{\star e(\varphi,\Gamma)}$. Closure property of Γ gives $(\star\varphi)^{\star e(\varphi,\Gamma)} \in \Gamma$, and by (11) we obtain $\star e(\varphi,\Gamma) = e(\star\varphi,\Gamma)$.

The proofs of (4) and (5) are similar to (3) with the rules $R_\neg(a)$ and $R_\neg(a)$ (see (10)) instead of $R_\star(a)$. The proofs of (6) and (7) are similar to (2) with the rules $R_\vee(a,b)$ and $R_\wedge(a,b)$ (see (10)) instead of $R_\circ(a,b)$. $\square$

Standard proofs show that the following Lemmas 2 and 3 hold. Notice that Lemma 1 is used in proof of Lemma 2.

Lemma 2. *For any nontrivial prime theory* Γ *and for any valuation* v_Γ *such that* $v_\Gamma(p) = e(p, \Gamma)$*, for any propositional variable* p*:* $v_\Gamma(\varphi) = e(\varphi, \Gamma)$*, for any* $\mathcal{L}_{\star\circ}$*-formula* φ*.*

Lemma 3. (*Lindenbaum*)*. For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formula* φ*: if* $\Gamma \nvdash \varphi$*, then there is a set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ^* *such that:* (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash \varphi$*, and* (3) Γ^* *is a nontrivial prime theory.*

Theorem 5. (*Completeness*)*. For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formula* φ*:* $\Gamma \models \varphi \Longrightarrow \Gamma \vdash \varphi$*.*

Proof. The proof proceeds by contraposition. Let $\Gamma \nvdash \varphi$. Then, by Lemma 3, there is a set of $\mathcal{L}_{\star\circ}$-formulas Γ^* such that: (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash \varphi$, and (3) Γ^* is a nontrivial prime theory. By Lemma 2, there is a valuation v_Γ such that $v_\Gamma(\psi) = \mathbf{T1}$, for any $\psi \in \Gamma$, and $v_\Gamma(\varphi) \neq \mathbf{T1}$. But then $\Gamma \not\models \varphi$. □

In the light of Theorems 4 and 5 the following Theorem 6 is obvious.

Theorem 6. (*Adequacy*)*. For any set of* $\mathcal{L}_{\star\circ}$*-formulas* Γ *and for any* $\mathcal{L}_{\star\circ}$*-formula* φ*:* $\Gamma \models \varphi \Longleftrightarrow \Gamma \vdash \varphi$*.*

Conclusion

In this paper, we have presented a general method (correspondence analysis) for axiomatizing LRA's unary and binary truth-functional extensions via natural deduction systems. The future work concerns an investigation of logics with LRA's connectives but with the other sets of designated values and constructing correspondence analysis for them.

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КОРРЕСПОНДЕНТСКИЙ АНАЛИЗ ДЛЯ ЛОГИКИ РАЦИОНАЛЬНОГО АГЕНТА

Я. И. Петрухин

Московский государственный университет имени М. В. Ломоносова, Москва, Россия
yaroslav.petrukhin@mail.ru

Рассматривается с теоретико-доказательной точки зрения логика рационального агента (LRA) Кубышкиной и Зайцева. В работе построено три системы натурального вывода для LRA, отличающиеся от аксиоматизации LRA, осуществлённой Кубышкиной и Зайцевым. Кроме того, сформулирован общий метод аксиоматизации с помощью натуральных исчислений расширений LRA любыми истинностно-функциональными одноместными и двухместными операторами. Этот метод есть не что иное, как описанный Коем и Таммингой корресподентский анализ, адаптированный в данном случае для LRA.

Ключевые слова: *многозначные логики, обобщённые истинностные значения, корреспондентский анализ, натуральное исчисление.*

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Сведения об авторе

Петрухин Ярослав Игоревич, студент кафедры логики философского факультета, Московский государственный университет имени М. В. Ломоносова, Москва, Россия; e-mail: yaroslav.petrukhin@mail.ru.